Conditions Under Which, in a Commutative GIS, Two 3-Element Sets Can Span the Same Assortment of GIS-Intervals; Notes on the Non-Commutative GIS In This Connection

David Lewin

Suppose A is a set of 3 pitches modulo the octave, using which we can span a just P5th, a just P4th, a just M3rd, a just m6th, a just m3rd and a just M6th. Suppose B is another such set. Can we conclude that B must be either a transposition or an inversion of A? Yes. Now suppose A is a set of three pitch classes from within a division of the octave into 51 equal units, and that if we use the members of A, we can span intervals of 9 units, 42 units, 21 units (in two different ways), and 30 units (also in two different ways). Suppose B is another such set. Can we conclude that B must be either a transposition or an inversion of A? Yes.

The situation can be generalized. If A and B are 3-element sets within any commutative GIS, and if B has the same GIS-interval content as A, then B must be either a GIS-transposition or a GIS-inversion of A. Furthermore, in any commutative GIS, any set must have the same GIS-interval content as any of its GIS-transposed or GIS-inverted forms.

In a non-commutative GIS, however, sets A and B may be related by GIS-transposition but have different GIS-interval contents. And in a non-commutative GIS, sets A and B may be related by GIS-inversion but have different GIS-interval contents. Finally, in a non-commutative GIS, *3-element* sets A and B may have the same GIS-interval content without being related either by GIS-transposition, or by GIS-inversion, or by some "interval-preserving transformation" of the GIS.

1. Setting the agenda.

1.1 By a *GIS*, I mean a Generalized Interval System in the sense in which I expounded the concept some years ago.[1] A GIS contains objects, and formal "intervals" between them, with certain stipulations about the behavior of said objects and intervals. The stipulations allow us to generalize a variety of ideas about traditional intervals among pitches or pitch classes, to ideas about formal GIS-intervals.

1.2 Among such ideas is the notion of tabulating the *[GIS]-interval content* of a finite set of GIS-objects. (We shall assume all things called "sets" in the sequel to be finite, even if the GIS at hand has infinitely many objects and intervals.) Formally, given a set A, its "[GIS]-interval content" is here defined to be a function f which tabulates the number of distinct ways in which each interval of the GIS system can be spanned within A. That is, for each interval i of the GIS, the function value f(i) is the number of different ways in which i can be spanned by members of A.[2]

1.2.1 As an example, let us consider the traditional GIS of pitch classes and pc intervals, and let us consider the set A which comprises the pitch classes B, C, and F. We write A = {B,C,F}. The GIS-interval content of A is shown in table 1.

[1]David Lewin, *Generalized Musical Intervals and Transformations* [henceforth *GMIT*] (New Haven: Yale University Press, 1987).

[2]I stress that the term "[GIS]-interval content" is here being *defined*, in exactly the manner described, for the present context. I introduced the term "interval(lic) content," thus defined, in *Journal of Music Theory* 4/1 (April 1960). My article (pages 98–101) was entitled, "The Intervallic Content of a Collection of Notes, Intervallic Relations between a Collection of Notes and its Complement, an Application to Schoenberg's Hexachordal Pieces." The "interval(lic) content" of a set A of notes is there defined as my "interval function from A to itself," the "interval function from A to B" having been defined in an earlier article, "Intervallic Relations Between Two Collections of Notes," *Journal of Music Theory* 3/2 (November 1959), 298–301.

While the 1960 article concerns itself only with the traditional GIS of "notes" (i.e. pcs in the mod 12 universe), the concept of "interval function" is generalized in *GMIT* so as to apply to any formal GIS structure, and subjected in that book to intense study. (The interested reader can consult IFUNC in the Index of the book.) The term "interval(lic) content" is not generalized in *GMIT*, though it could easily have been, as it is in the present article.

Table 1

interval i:	0	1	2	3	4	5	6	7	8	9	10	11
value f(i):	3	1	0	0	0	1	2	1	0	0	0	1

Some comments are in order.

(1) The top row of the table lists the twelve *directed pc intervals* of the GIS, numbered 0 through 11, rather than 6 "interval-classes" ("ic1" through "ic6").

(2) Even though the value of f(−i) is the same as the value of f(i) on the table, the conceptual distinction between the values is significant: "f(1)=1" signifies that there is 1 way to span a directed interval of 1 using members of the set (namely, from B to C), while "f(11)=1" signifies that there is 1 way to span a directed interval of 11 using members of the set (namely, from C to B).

(3) "f(0) = 3" signifies that there are three distinct ways in which the member objects of set A can span a unison, an interval 0 of the GIS. "f(6) = 2" signifies that there are two distinct ways in which the member objects of set A can span a tritone, an interval 6 of the GIS—either from B to F or from F to B.

These features of the GIS-interval content function distinguish it from Forte's "interval vector," which it otherwise resembles.[3]

1.3 A GIS structure enables us to construct, for any set therein, the function of section (1.2), its formal "GIS-interval content." We can then ask: under what conditions can two different sets in

[3]Forte's interval vector, basically, counts types of 2-note subsets of set A, not ways of spanning intervals within set A. So it inspects 6 things (2-note set types), rather than 12 things (directed pc intervals). Forte's vector does not count unisons, which are not 2-note sets. His vector also counts one tritone-dyad (e.g. {B,F}), where the function of table 1 counts two ways of spanning a directed tritone-interval, either from B to F, or from F to B.

Allen Forte defines and discusses his "interval vector" in *The Structure of Atonal Music* (New Haven: Yale University Press, 1973), 15*ff.*

There is a substantial literature on generalizing Forte's "interval vector," as well as my "interval function." A later note will take the matter up.

a GIS have the same GIS-interval content? The present study answers that question definitively for sets of cardinality 3 in the most general GIS of the sort called "commutative." (The term will be defined formally later on in this paper.)

1.4 In the traditional GIS of pitch classes and pitch-class intervals, the question just asked has been answered not just for sets of cardinality 3, but for sets of all cardinalities:

(a) In that GIS, sets that are (GIS)-transpositions or (GIS)-inversions, each of the other, must have the same (GIS)-interval content.

(b) In that GIS, some pairs of sets have the same (GIS)-interval content but are not related by (GIS)-transposition or (GIS)-inversion. Here, I shall call such pairs "GISZ-related."[4]

(c) In that GIS, there are no GISZ-related pairs of cardinality 3.

(d) There are, however, GISZ-related pairs of cardinality 4, of cardinality 5, and so forth through cardinality 8. These GISZ-pairs have been tabulated.[5]

[4]The symbol "GISZ" was chosen to impress upon the reader the completely formal and absolutely specific character of the term in this context: GISZ-related sets (1) must be sets within a specified GIS, (2) must have the same formal GIS-interval content therein, and (3) must not be related therein either by formal GIS-transposition or by formal GIS-inversion. (Those transformations will be described presently.)

In the traditional mod 12 GIS of pcs and directed pc intervals, it so happens that two sets (of any cardinality) are GISZ-related in the present sense if and only if they are "Z-related" in the sense of Allen Forte (*The Structure of Atonal Music*, 21–24). Forte's Z-relation has been generalized and extended in many ways. The GISZ construction is one particular way. Other generalizations and extensions are discussed in *GMIT* (103–122). Yet other important extensions have been explored by Robert Morris, the locus classicus being his "Set Groups, Complementation, and Mappings among Pitch-Class Sets," *Journal of Music Theory* 26/1 (1982): 101–44.

However, the point of the present paper is not to explore any other extensions—or even properties—of Forte's Z-relation. The point is rather to investigate aspects of the behavior of certain 3-element sets within a GIS. Hence, in speaking of the GISZ-relation, I have chosen a name that ostentatiously points to the presence and roles, in determining that relation, of a specified GIS, the GIS-interval content, GIS-transposition, and GIS-inversion.

1.5 As a result of (1.4c) above, another question arises: is the absence of GISZ-related 3-element sets, in the traditional GIS under discussion, a special feature of that *particular* GIS? Or is it a necessary feature of *any* GIS whatsoever? In any GIS, we can always speak of formal GIS-transposition, formal GIS-inversion, and a formal GIS-interval-content function, so we can always inquire after 3-element sets that are "GISZ-related" in the particular sense defined by (1.4b) above. The present paper will show that the absence of GISZ-related 3-element sets is *not* a necessary feature of *any* GIS whatsoever, but that it *is* indeed a necessary feature of any *commutative* GIS.

After we have shown that a commutative GIS cannot have GISZ-related 3-element sets, we shall discuss the problems that prevent us from generalizing our result to the non-commutative case.

1.6 In order to carry out our agenda, we shall first review the basic definitions and characteristic features of a GIS.

2. GIS structure; formalities.

The following summary is taken from an earlier summary of mine.[6]

2.1 To manifest a Generalized Interval System we require:
a family of formal *objects* s, t, ... ,
a mathematical group of formal *intervals* i, j, ... , and
a *function int* which assigns to any ordered pair <s,t> of objects a value int(s,t) within the group of intervals.

If int(s,t) = i, one says that "i is the interval from s to t" within the GIS.

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[5]Forte, *The Structure of Atonal Music*, Appendix 1, 179–81. A set that is Z-related to some other appears on this table with a Z in its look-up number, e.g. "5–Z17." As observed in note 4, our GISZ-relation coincides with his Z-relation in this particular GIS.

[6]David Lewin, "Generalized Interval Systems for Babbitt's Lists, and for Schoenberg's String Trio," *Music Theory Spectrum* 17/1 (Spring 1995): 81–118. The material used here comes from pages 82–83 of that article.

We further require that for every three objects r, s, and t, the interval from r to s, when combined in the group with the interval from s to t, yields the interval from r to t. Symbolically, the requirement is that int(r,s)int(s,t) = int(r,t).

Finally, we require that for each given object s and each given interval i, there be a unique object t which lies the given interval from the given object—i.e. which satisfies the equation int(s,t) = i.

A GIS is called *commutative* or *non-commutative* depending on its group of intervals. That group is *commutative* when ij = ji for all intervals i and j, that is, when i-combined-with-j in the group yields the same result as does j-combined-with-i.

The following synopsis is also taken from an earlier work of mine.[7] It defines formal GIS-transposition for any GIS.

2.2 In an abstract GIS, the operation of *GIS-transposition by interval i* is well-defined by the formula int(s,Ti(s)) = i. That is, for any object s, the Ti-transform of s is that unique object Ti(s) which lies the interval i from s. The GIS-transposition operations form a mathematical group which is anti-isomorphic to the group of intervals. Specifically, TiTj = T(ji); i.e. the i-GIS-transpose of the j-GIS-transpose (of any object) is the (ji)-GIS-transpose (of that object).

When there is no danger of confusion, we shall write "transposition" for "GIS-transposition."

An *interval-preserving operation* upon the objects of a GIS is a transformation P satisfying the formula int(P(s),P(t)) = int(s,t). That is, the interval between the P-transforms of any two objects is the same as the interval between the objects themselves. The interval-preserving operations form a group that can be proved isomorphic to the group of intervals.

If (the group of intervals for) a GIS is commutative, the GIS-transpositions are exactly the interval-preserving operations: a transformation preserves intervals if and only if it is a GIS-transposition. If a GIS is non-commutative, there will be GIS-transpositions that do not preserve intervals, and there will be interval-preserving operations that are not GIS-transpositions.

[7]Ibid., page 100.

The following summary is taken from my earlier work as well.[8] It defines formal GIS-inversion, for any GIS. (Some of the symbols are changed for present purposes.)

2.3 In an abstract GIS, given any objects y and v (where v may be the same object as y), the operation I of *y/v GIS-inversion* is well-defined by the formula int(v,I(s)) = int(s,y). The formula expresses a pertinent intuition: given any object s, its inverted transform I(s) lies intervallically in relation to v, exactly as y lies intervallically in relation to s. Formally, given some object s, set i = int(s,y), and then find the unique object t which lies the interval i from v. The t so found satisfies the equation int(v,t) = i = int(s,y); t is taken to be the pertinent GIS-inverted image I(s) of s.

When there is no danger of confusion, we shall write "inversion" for "GIS-inversion."

An *interval-reversing transformation* on the objects of a GIS is a transformation R satisfying the formula int(R(s),R(t)) = int(t,s). That is, the interval between the R-transforms of any two objects is the same as the interval between the objects themselves in reverse order.

If a GIS is commutative, the GIS-inversions are exactly the interval-reversing transformations: a transformation reverses intervals if and only if it is a GIS-inversion. If a GIS is non-commutative, GIS-inversions will not reverse intervals in all cases (for all s and t). Indeed, a non-commutative GIS possesses no interval-reversing transformations whatsoever.

3. In any *commutative* GIS, (1.4abc) generalize completely.

Throughout part 3, we shall assume a commutative GIS to be fixed.

3.1 In dealing with a general commutative group, it is customary to use symbols associated with addition. Thus, in our commutative GIS, the interval i and the interval j combine to form the interval i + j. Since the group is commutative, i + j = j + i.

[8]*GMIT*, 51 and 58.

So the intervals j and i (in that order) combine to form the same sum-interval as do the intervals i and j (in that order). The group of intervals has a unique identity interval 0.

The zero interval may be thought of as a generalized "unison" interval: it combines with any interval i to form that very interval i. Every interval i has a unique inverse interval, denoted "–i"; –i is that unique interval which combines with i to yield the identity 0. (Example in the traditional pitch-class GIS: –4 = 8, since if one proceeds by 4 pc semitones, and then proceeds further by 8 pc semitones, one traverses 0 pc semitones net.)

3.2 We now proceed to show that (1.4a) generalizes for our commutative GIS. That is, *in a commutative GIS, every GIS-transposed or GIS-inverted form of a given object-set has the same GIS-interval content as that given set.*

Proof: Let A be a set of distinct objects x, y, z, ... Let T be a transposition-operation of the GIS. The GIS-transposed set T(A) = B then comprises the distinct objects T(x), T(y), T(z) ... .[9]

Suppose the interval i is spanned between objects s and t of set A. Then, since GIS-transposition is interval-preserving in our commutative (!) GIS, the interval i will also be spanned between objects T(s) and T(t) of the transposed set T(A) = B. So *the objects of B span at least as many intervals of i, as do the objects of A itself.*

But A is also a GIS-transposed form of B: A = T'(B), where T' is the inverse transposition to T. By the argument of the preceding paragraph, applied to B and T'(B) = A, we conclude that *the objects of A span at least as many intervals of i, as do the objects of B.*

From the italicized observations in the preceding two paragraphs, one infers that *the objects of B span exactly as many intervals of i as do the objects of A.* This being the case for any sample interval i, we conclude that *B has the same formal GIS-interval content, as does A.*

[9]The transposed objects will be distinct because T is "1-to-1 and onto." The formalities of that observation are proved on pages 46–47 of *GMIT* (§3.4.2).

If B = I(A) is a GIS-inverted form of A, the argument is almost the same. We need only observe that all inversion operations, in our commutative (!) GIS, are interval-reversing. Then the interval i will be spanned between objects s and t of set A, if and only if the interval i is spanned between objects I(t) and I(s) of set B.[10] The rest of the argument proceeds exactly as for GIS-transposition.

3.3 We can now generalize (1.4b), in our general commutative GIS. That is, we can define two sets-of-objects to be *GISZ-related*, if they have the same GIS-interval content but are not GIS-transposed or GIS-inverted forms, each of the other. To be sure, we could always make this definition anyway, but the restriction, that GISZ-related sets should not be related by some T or I, makes sense here only in light of (3.2).[11]

3.4 Now we are ready for the first substantial result, generalizing (1.4c): *in any commutative (!) GIS, there are no GISZ-related sets of cardinality 3*. That is to say, *if two 3-element sets of a commutative GIS have the same GIS-interval function, they must be related by GIS-transposition or GIS-inversion*.

[10]The demonstration, that inversion-transformations as defined are 1-to-1 and onto, is sketched on page 51 of *GMIT* (§3.5.1).

[11]Once again let me point out to the reader that I am not proposing here to discuss Forte's Z-relation as such, nor any other of its generalizations. Nor am I proposing to discuss other generalizations of my "interval function" or IFUNC (or injection function, or canonical group, or...). I am proposing to discuss (finite) sets within (finite or infinite) GIS-structures, and in the future I am going to focus particularly on 3-element sets within GIS-structures. When I ask of sets A and B, if they are GISZ-related, I am supposing there is a well-defined GIS at hand (and not some more general transformational system, perhaps involving a "canonical group" of operations, or whatever). I am supposing that A and B are sets within that specified GIS. I am then asking *only* (1) and (2) following: (1) do A and B have the same formal GIS-interval content as defined in the present paper? (2) Are A and B formal GIS-transpositions (or later GIS-interval-preserving transformations) or formal GIS-inversions, each of the other? If the answer to (1) is yes, and the answer to (2) is no, then A and B are GISZ-related, by the definition of "GISZ." Otherwise A and B are not GISZ-related (by the same definition).

The result will show that (1.4c) was not just a special property of the traditional pitch-class GIS; it is rather a property *necessarily* enjoyed by *any* commutative GIS. (Later on, we shall show that it is *not* necessarily enjoyed even more generally, in a reasonable sense, by a general *non-commutative* GIS. But for the time being, we are restricting our attention to commutative GIS structures.)

3.4.1 To prove the assertion of 3.4, we shall suppose 3-element sets A = {x, y, z} and B = {u, v, w} which have the same GIS-interval content; we will then prove the following assertion: there exists a GIS-transposition T, or a GIS-inversion I, such that B is the T-or-I transform of A.

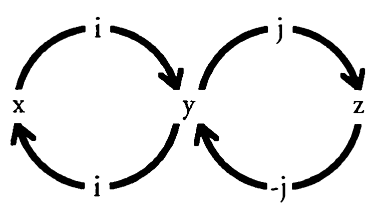
3.5.1 Theorem: the assertion of (3.4.1) is true if the 3-element set A = {x,y,z} spans some non-zero interval i in two different ways, and i + i = 0.

(This would be the case, for example, in the traditional pitch-class GIS for the pcset {B,C,F}, which spans the interval i = 6 from B to F, and also from F to B, with 6 + 6 = 0.)

Reshuffling the labels x, y, and z as appropriate, we can assume that i is the interval from x to y.

Now in any GIS, int(t,s) is the inverse of int(s,t).[12] For the particular x and y under present consideration, then, the interval from y to x is −i, the inverse of i. And for our particular i here, i + i = 0; hence −i is i itself. So i is not only the interval from x to y, it is also the interval from y to x. We can then partially diagram the intervallic structure of A as in Figure 1.

Figure 1

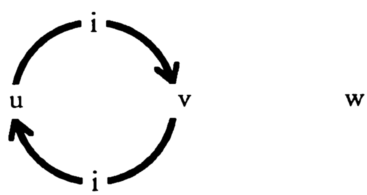


[12]GMIT, 26 (§2.3.2).

In the arrangement of Figure 1 the interval j, from y to z, cannot be the same interval as i. For i is the interval from y to x; i cannot also be the interval from y to z, since z is distinct from x. (Given y and the interval i, there is a *unique* element which lies the interval i from y; that unique element cannot be both x and z.)

3-element set B = {u,v,w} is assumed by (3.4.1) to have the same GIS-interval content as set A, so set B spans the interval i in some way. By reshuffling the labels u, v, and w for the members of B, we can assume that the situation of Figure 2 obtains.

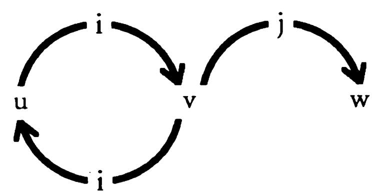
Figure 2



Now B, which has the same interval content as A, must span the interval j in some way, and j is not the same as i. Therefore element w of Figure 2 must be involved somehow in a spanning of j within B: either j = int(v,w) or j = int(u,w) or j = int(w,v) or j = int(w,u). We shall call these four alternatives "Case 1," "Case 2," "Case 3," and "Case 4."

Case 1: j = int(v,w). Figure 2 then assumes the aspect of Figure 3.

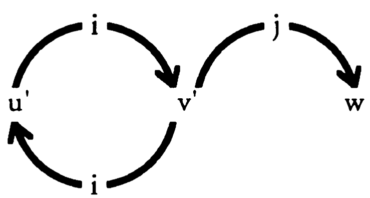
Figure 3



In the orderings of Figure 1 and Figure 3, we see that x-to-y-to-z proceeds via intervals i and j; u-to-v-to-w also proceeds by intervals i and j. It follows that *the 3-element set B is a transposition of the 3-element set A.* To see this, let n = int(x,u). Then int(y,v) = int(y,x) + int(x,u) + int(u,v) = i + n + i = i + i + n (since the GIS is commutative!) = 0 + n = n. In like fashion, int(z,w) = int(z,y) + int(y,v) + int(v,w) = –j + n + j = n (since the GIS is commutative). In sum, int(x,u) = n; int(y,v) = n, and int(z,w) = n. Thus set B is the n-transpose of set A. *If Case 1 obtains, theorem (3.5.1) is true.*

Case 2: j = int(u,w). We relabel the elements of B as u', v', and w, setting u' = v and v' = u. Figure 2 then assumes the aspect of Figure 4.

Figure 4



In Figure 4, int(u',v') = int(v,u) = i, int(v',u') = int(u,u) = i, and int(v',w) = int(u,w) = j (by the supposition of Case 2).

Just as we used Figures 1 and 3 earlier (in Case 1), to show that set B was transposition-by-n of set A, where n was int(x,u), so we can use Figures 1 and 4 now (in Case 2), to show that set B, in this case, is transposition-by-n' of set A, where n' is int(x,u') = int(x,v). *If Case 2 obtains, theorem (3.5.1) is true.*

Case 3: j = int(w,v). Then int(v,w) = –j. We can now recast Figure 2 in the form of Figure 5.

Let I be the operation of y/v inversion. The defining formula of that operation, in (2.3) above, is int(v,I(s)) = int(s,y): for any sample object s, the inverted object I(s) lies the same interval from v as y lies from the sample object s.

Take the sample s to be x. The formula then defines I(x) by the equation int(v,I(x)) = int(x,y). And int(x,y) is i (as we see on

Figure 5

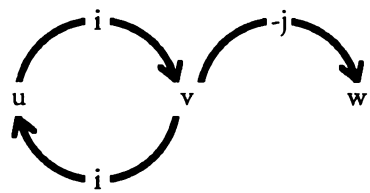


Figure 1). So int(v,I(x)) = i; I(x) is that unique object which lies the interval i from v. Inspecting Figure 5, we see that I(x) = u.

Next, take the sample s of the formula to be y. The formula then defines I(y) by the equation int(v,I(y)) = int(y,y). And int(y,y) is 0, the unison interval. So int(v,I(y)) = 0; I(y) is that unique object which lies the unison interval from v. That is, I(y) = v.

Next, take the sample s of the formula to be z. The formula then defines I(z) by the equation int(v,I(z)) = int(z,y). And int(z,y) is –j (as we see on Figure 1). So int(v,I(z)) = –j; I(z) is that unique object which lies the interval –j from v. Inspecting Figure 5, we see that I(z) = w.

We have seen: I(x) = u, I(y) = v, I(z) = w. Hence 3-element set B is the I-inversion of set A. *If Case 3 obtains, theorem (3.5.1) is true.*

Case 4: j = int(w,u). As in Case 2, we exchange the roles of u and v by setting u' = v, v' = u. Case 4 then reduces to Case 3. B is the I-inversion of A, where I is the inversion that maps y to v' (that is, y to u).

This finishes all four possible Cases, so theorem (3.5.1) is fully proved: *If 3-element sets A and B, in a commutative (!) GIS, have the same GIS-interval content, and if set A contains an interval which is its own inverse interval, then sets A and B are transpositions or inversions, each of the other.*

3.5.2 Theorem: the assertion of (3.4.1) is true if the 3-element set A = {x,y,z} spans some interval i in two different ways, and i is *not* its own inverse.

(This would be the case, e.g., in the traditional pitch-class GIS for the pcset A = {C,D,E}: A spans the interval i = 2 in two different ways, and interval 2 is not its own inverse.)

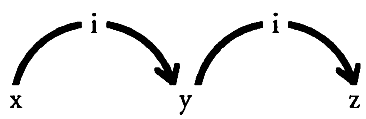
Shuffling labels as necessary for the objects of A = {x,y,z}, we can suppose that i = int(x,y). Then we do *not* have int(y,x) = i, for otherwise i would be its own inverse, contrary to present assumption. Since there is some other way of spanning i within A (other than from from-x-to-y), this other way must then somehow involve element z. That is, either int(x,z) = i, or int(y,z) = i, or int (z,x) = i, or int(z,y) = i.

But int(x,z) = i is impossible: y is the *unique* element lying the interval i from x, and the element z is distinct from y. Likewise int(z,y) = i is impossible. If it were the case we would have int(y,z) = –i. But int(y,x) = –i, and z is distinct from x.

So we must have either int(y,z) = i, or int(z,x) = i.

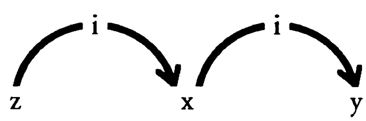
Case 1: int(y,z) = i. We may then diagram the intervallic structure of set A as in Figure 6.

Figure 6



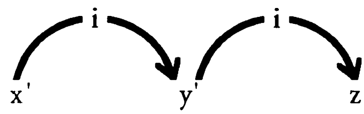
Case 2: int(z,x) = i. We may then diagram the intervallic structure of set A as in Figure 7.

Figure 7



Setting x' = z, y' = x, and z' = y, we can recast Figure 7 in the form of Figure 8.

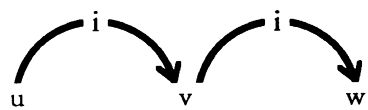
Figure 8



So, by shuffling the labels for the elements of set A, we may assume the configuration of Figure 6.

Since 3-element set B = {u,v,w} has the same interval content as set A, we may reshuffle the labels for *its* elements in just the same way, and suppose the configuration of Figure 9.

Figure 9



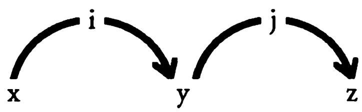
Using Figures 6 and 9, we can then show that set B is the n-transpose of set A, where n is the interval from x to u. This finishes the proof for theorem (3.5.2). We have proved: *if 3-element sets A and B, in a commutative (!) GIS, have the same GIS-interval content, and if set A spans in two different ways an interval i which is not its own inverse interval, then sets A and B are transpositions or inversions, each of the other.*

3.5.3 Theorem: the assertion of (3.4.1) is true if the 3-element set A = {x,y,z} spans *no* non-zero interval i in two different ways, so that the six non-zero intervals of set A are distinct.

(This would be the case, e.g., in the traditional pitch-class GIS for the pcset A = {C, D, F}: A spans one each of the six distinct intervals i = 2, 3, 5, 7, 9, and 10.)

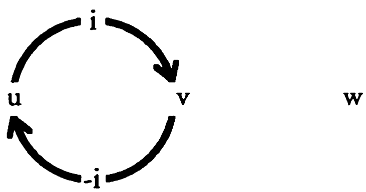
For the set A, we may suppose the configuration of Figure 10, where the intervals i and j, by supposition, are different.

Figure 10



Since interval i is spanned in some way by the member elements of 3-element set B = {u,v,w}, we may reshuffle the labels for those elements as necessary, and suppose that i is the interval from u to v, as in Figure 11.

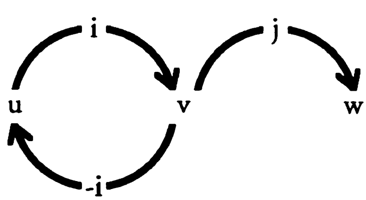
Figure 11



Minus-i appears on the figure as the interval from v to u. By supposition, the intervals i, –i, and j are all distinct. Now interval j is spanned in some way by member elements of the 3-element set B = {u,v,w}, and interval j (being distinct from i and from –i) is not spanned from u to v, nor from v to u. Hence one of the elements involved in spanning j is w. Thus either j = int(v,w), or j = int(w,u), or j = int(w,v), or j = int(u,w). We go through the four cases in turn.

Case 1: j = int(v,w). The situation of Figure 12 obtains.

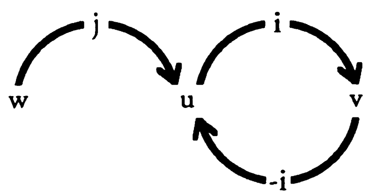
Figure 12



As in (3.5.1) earlier for the analogous case, we consider transposition T by the interval n that extends from x to u. Using figures 10 and 12, we infer that int(y,v) = int(y,x) + int(x,u) + int(u,v) = –i + n + i = n (since the GIS is commutative). Following that, we infer int(z,w) = int(z,y) + int(y,v) + int(v,w) = –j + n + j = n (since the GIS is commutative). So we have int(x,u) = n, int(y,v) = n, and int(z,w) = n. u = Tn(x), v = Tn(y), and v = Tn(z); 3-element set B is the Tn-transpose of 3-element set A. *In Case 1, theorem (3.5.3) is proved.*

Case 2: j = int(w,u). The situation of Figure 13 obtains.

Figure 13



Let I be the operation of y/u inversion. The defining formula of that operation, as in (2.3) above, is int(u,I(s)) = int(s,y): for any sample object s, the inverted object I(s) lies the same interval from u, as y lies from the sample object s.

Taking our sample s to be y, we have int(u,I(y)) = int(y,y), the identity ("unison") interval. So I(y) lies a "unison" from u, and u is the I-inversion of y.

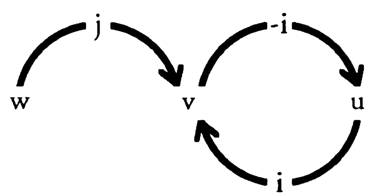
Taking our sample s to be x, the formula tells us that int(u,I(x)) = int(x,y). Inspecting Figure 10, we see that int(x,y) = i. So I(x) lies the interval i from u; inspecting Figure 13 we see that I(x) = v.

Taking our sample x to be z, the formula tells us that int(u,I(z)) = int(z,y). Figure 10 tells us that int(z,y) = –j. So I(z) lies the interval –j from u; inspecting Figure 13, we see that I(z) = w.

In sum: I(x) = v, I(y) = u, and I(z) = w. Thus 3-element set B is the I-inversion of 3-element set A. *In Case 2, theorem (3.5.3) is proved.*

Case 3: j = int(w,v). We shall show that *this case cannot occur under present assumptions, in particular the assumption that our GIS is commutative (!).* If the case did occur, it would entail the set-up of Figure 14.

Figure 14



From Figure 10, we can deduce all six non-zero intervals of set A. They are namely i, –i, j, –j, i + j, and –j –i.[13] By an assumption of (3.5.3), these six intervals are distinct. Analogous inspection of Figure 14 yields the six distinct non-zero intervals of set B: they are i, –i, j, –j, j – i, and i – j.

Since sets A and B are assumed to have the same interval content, the interval i + j of set A must be some interval of set B, whose distinct non-zero intervals are i, –i, j, –j, j – i, and i – j. The interval i + j of set A cannot be i, –i, j, or –j. (i, –i, j, and –j are intervals of A presumed by (3.5.3) to be distinct from the interval i + j of set A). Hence, inspecting the list of intervals for B, we see that, since i + j cannot be i, –i, j, or –j, interval i + j must be either j – i or i – j.

If i + j = j – i, then we can cancel the j's in the equation to obtain i = –i. But that is contrary to the assumption of (3.5.3).[14]

[13]Inspecting Figure 10, we can see that i is spanned from x to y, –i from y to x, j from y to z, –j from z to y, i + j from x to z, and –j – i from z to x.

[14]This part of the proof will not go through in a *non*-commutative GIS. To see this, we shall work out the cancellation process in more detail. Knowing that i + j = j – i here, we wish to "cancel the j's in the equation to obtain i = –i." To do this, we subtract j from the right on both sides of the known equation. That is, given i + j = j – i, we infer that (i + j) – j = (j – i) – j. Hence (redrawing the parentheses, which is okay in any group) we infer that i + (j – j) = j – i – j; and thence we infer that i = j – i – j. In our *commutative* GIS, we have j – i = –i + j, so we can infer that i = (j – i) – j = (–i + j) – j = –i + (j – j) = –i. The inference

Likewise, if i + j = i − j, we can infer j = −j. But that is contrary to the assumption of (3.5.3).

Thus the supposition of Case 3 leads to a contradiction. *Case 3 cannot happen.*

In similar fashion, we show that *Case 4 cannot happen*: it is not the case that j = int(u,w).[15]

Therefore, either Case 1 or Case 2 must obtain. 3-element set B is accordingly a GIS-transposition or a GIS-inversion of set A. *If 3-element sets A and B, in a commutative (!) GIS, have the same GIS-interval content, and if set A spans no non-zero interval in two different ways, then sets A and B are transpositions or inversions, each of the other.*

(3.5.1), (3.5.2), and (3.5.3) exhaust all possible cases as regards the interval content of the 3-element set A = {x,y,z}. Accordingly the assertion of (3.4.1) is now established in any of those cases.

And so the assertion of (3.4) is established: *in any commutative (!) GIS, there are no GISZ-related sets of cardinality 3.* That is to say, *if two 3-element sets of a commutative GIS have the same GIS-interval function, they must be related by GIS-transposition or GIS-inversion.*

We also showed in (3.1) that in any commutative (!) GIS, GIS-transposed or GIS-inverted sets must share the same GIS-interval content. So the entire agenda promised by the title of part 3 is now accomplished: *in any commutative GIS, (1.4abc) generalize completely.*

4. Discussion of the non-commutative case.

4.1.1 In a non-commutative group, it is conventional to use symbolic multiplication as the group combination. The identity element of the group is conventionally denoted by the letter e: ei = ie = i for every group member i. The inverse element of i

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that i = −i is a crucial step in the present proof. But, in a *non*-commutative GIS, we cannot arrive at that inference, because it is not necessarily the case that j − i is equal to −i + j.

[15]Again, we lean heavily here on the presumption that our GIS is commutative.

within the group is conventionally denoted as "i-inverse," written as i with the superscript minus-one, thus: i^{-1}. So $ii^{-1} = i^{-1}i = e$.

4.1.2 It will be useful to note the following: any group G can be used as the family of formal objects for a GIS, if we take G itself as a group of formal intervals, and define int(s,t) to be $s^{-1}t$. We have $(r^{-1}s)(s^{-1}t) = r^{-1}t$; thus int(r,s)int(s,t) = int(r,t), and the first requirement of (2.1) is satisfied.

The second requirement of (2.1) is also satisfied: given an object s (in G) and an interval i (in G), suppose that t satisfies the equation int(s,t) = i. That is, suppose that $s^{-1}t = i$. We can multiply both sides of that equation on the left by s, and infer that t = si. And in fact t = si does satisfy the equation: $s^{-1}si = i$. So t is a ***unique*** element of the GIS satisfying int(s,t) = i.[16]

4.1.3 In the GIS of (4.1.2), GIS-transposition-by-x is given by the formula T(s) = sx.

The elements x of the group can also be used to define the interval-preserving transformations of the group: a typical interval-preserving operation is given by the formula P(s) = xs.

A typical GIS-inversion operation of the GIS is given by the formula $I(s) = xs^{-1}y$, where x and y are elements of the group.[17]

(4.2) following sets down, for convenient reference, the generalization of (1.4abc) which we have just seen is the case in any commutative GIS:

4.2 (a) In any commutative GIS, sets that are GIS-transpositions or GIS-inversions, each of the other, must have the same GIS-interval content.

[16] In light of (4.1.2), we can regard the *mechanics* of GIS theory as a part of traditional mathematical group theory. But as a *model* for *musical* systems, a formal GIS does well to separate the role of its *objects* (like pitch classes), from the role of its *intervals* (like pc intervals). The objects do not exhibit algebraic behavior in themselves; the intervals do.

I discuss related matters in *GMIT*, 31–32.

[17] Relevant material will be found in *GMIT*, 47–52

(b) In any commutative GIS, some pairs of sets may have the same GIS-interval content without being related by GIS-transposition or GIS-inversion. Such pairs are here being called "GISZ-related."

(c) In any commutative GIS, there are no GISZ-related pairs of cardinality 3. Any 3-element set is thus completely determined, up to GIS-transposition and GIS-inversion, by its GIS-interval content.

4.3.1 In trying to generalize (4.2) to a non-commutative GIS, our first problem is that (4.2a) need no longer be the case. As noted in the third paragraph of (2.2) above, there will be at least one GIS-transposition operation T that does not preserve intervals, so we cannot conclude in all cases that a set A must have the same interval content as the set B = T(A).[18]

4.3.2 I will give here, in fact, a specific example of a GIS, a (2-element) set A, and a (2-element) set B, such that set B is a GIS-transposition of set A, but does not have the same GIS-interval content as A. Readers willing to take this on faith can move ahead to (4.4.1).

We take as the members of a group G the six elements e (identity), a, a^2, b, ab, and a^2b, subject to the requirements that $a^3 = e$, $b^2 = e$, $ba = a^2b$, and $ba^2 = ab$. The "multiplication table" shown in Table 2 confirms that this is a group; the product of any two elements is another of the same six elements.

Table 2

	e	a	a^2	b	ab	a^2b
e	e	a	a^2	b	ab	a^2b
a	a	a^2	e	ab	a^2b	b
a^2	a^2	e	a	a^2b	b	ab
b	b	a^2b	ab	e	a^2	a
ab	ab	b	a^2b	a	e	a^2
a^2b	a^2b	ab	b	a^2	a	e

18 We can review in particular how the proof of (3.2) relied explicitly upon our assumption that the GIS under consideration there was commutative.

We consider G to be a GIS in the manner of (4.1.2) above. We take as "set A" in this GIS the two-element family comprising objects x = a and y = b.

We take as transposition T the operation of GIS-transposition by interval a. The operation is defined (as in (4.1.3) above) by the formula T(s) = sa.

The set B = T(A) then comprises the two objects u = T(a) = aa = a^2, and v = T(b) = ba = a^2b.

We compute the two non-identity intervals of set A: int(x,y) = $x^{-1}y = a^{-1}b = a^2b$; int(y,x) is the inverse interval to int(x,y), the inverse of a^2b, which is ba = a^2b itself. Thus the two non-identity intervals of set A are both a^2b.

We compute the two non-identity intervals of set B: int(u,v) = $u^{-1}v = (a^2)^{-1}b$ = ab; int(v,u) is the inverse interval to int(u,v), the inverse of ab, which is ba^2 = ab itself. Thus the two non-identity intervals of set B are both ab.

In sum, B is a transposition of A, but B (whose intervals are e, e, ab, and ab) does not have the same interval content as A (whose intervals are e, e, a^2b, and a^2b).

4.3.3 We can remove the problem just discussed by considering *interval-preserving transformations* rather than GIS-transpositions. In a commutative GIS, the two sorts of operations are exactly the same. And in any non-commutative GIS, if P is an interval-preserving transformation and set B is the P-transform of set A, then B must have the same interval content as A.[19]

Substituting interval-preserving transformations for GIS-transpositions, we can now revise (4.2a) to ask: is (4.4.1) the case?

[19]To see this, we reason as follows. Let i be any sample interval, and suppose interval i is spanned between members x and y of set A. Then, since int(P(x),P(y)) = int(x,y), i is spanned between members P(x) and P(y) of set B. Conversely, if i is spanned between members P(x) and P(y) of set B, i must be spanned between members x and y of set A. Consequently, the number of ways that i is spanned by members of set A is exactly the same as the number of ways that i is spanned by members of set B. This being the case for any sample interval i, we conclude that B has the same interval content as A.

4.4.1 (a) Is it true that in any non-commutative GIS, sets that are related by interval-preserving transformation or by GIS-inversion must have the same GIS-interval content?

(b) Is it true that in any non-commutative GIS, some pairs of sets may have the same GIS-interval content without being related by interval-preserving transformation or GIS-inversion? If so, we might wish to call such pairs "GISZ-related."

(c) Is it true that in any non-commutative GIS, there are no GISZ-related pairs (in the above sense) of cardinality 3?

4.4.2 No, (4.4.1) is not the case. Indeed, (4.4.1a) is still not the case. As noted in the third paragraph of (2.3) above, GIS-inversion operations will not be interval-reversing—indeed there will be no interval-reversing transformations at all in our non-commutative GIS. So we cannot conclude in all cases that a set A must have the same GIS-interval content as a GIS-inverted set of form B = I(A).[20]

4.4.3 I will give here, in fact, a specific example of a GIS, a set A, and a set B, such that set B is a GIS-inversion of set A, but does not have the same GIS-interval content as A. Readers willing to take this on faith can move ahead to (4.5.1).

We consider the group G already used in the example of (4.3.2): the six elements of G are e (identity), a, a^2, b, ab, and a^2b, subject to the requirements that $a^3 = e$, $b^2 = e$, $ba = a^2b$, and $ba^2 = ab$.

As in (4.3.2), we consider G to be a GIS, and we take as "set A" in this GIS the two-element family comprising objects x = a and y = b.

We take as inversion I the operation of e/e inversion. The operation is defined (as in (2.3) above) by the formula int(e,I(s)) = int(s,e). In our present GIS, the formula means that $e^{-1}I(s) = s^{-1}e$. That is, $I(s) = s^{-1}$. The set B = I(A) then comprises the two objects $u = I(a) = a^{-1} = a^2$, and $v = I(b) = b^{-1} = b$.

[20]The proof of (3.2), toward its end, relied explicitly upon our assumption that the GIS under consideration there was commutative, so that we could treat inversion operations in all generality as interval-reversing.

We have already computed the two non-identity intervals of set A; we do so again here: $\mathrm{int}(x,y) = x^{-1}y = a^{-1}b = a^2b$; $\mathrm{int}(y,x)$ is the inverse interval to $\mathrm{int}(x,y)$, the inverse of a^2b, which is $ba = a^2b$ itself. Thus the two non-identity intervals of set A are both a^2b.

We compute the two non-identity intervals of our present set B: $\mathrm{int}(u,v) = u^{-1}v = (a^2)^{-1}b = ab$; $\mathrm{int}(v,u)$ is the inverse interval to $\mathrm{int}(u,v)$, the inverse of ab, which is $ba^2 = ab$ itself. Thus the two non-identity intervals of set B are both ab.

In sum, B is an inversion of A, but B (whose intervals are e, e, ab, and ab) does not have the same interval content as A (whose intervals are e, e, a^2b, and a^2b).

4.5.1 Finally, (4.2c) is not true in every non-commutative GIS structure. That is, there exist GIS structures in which two 3-element sets A and B can have the same interval content, without being related by some interval-preserving operation (or by some transposition), or by some inversion operation.

4.5.2 I will give here, in fact, a specific example of a GIS, a 3-element set A, and a 3-element set B, such that B has the same interval content as A, but is neither an interval-preserving transform of A (nor a transposition of A) nor an inversion of A. To the extent that the category of "GISZ-sets" is at all useful, then, A and B could be called "GISZ-related trichords."

The demonstration that ensues here requires a certain level of experience in handling the vocabulary and techniques of group theory. Inexperienced readers may choose either to end their reading here or to consult with more mathematically experienced friends for confirmation. The demonstration now begins.

4.5.2.1 Take as a group G the group with two generators, i and j, subject to the rule that $ji = i^{-1}j$. From that rule it follows that $ji^{-1} = ij$, and that $j^2i = ij^2$. From these observations the following formula can be deduced:

FORMULA: for any integers b and c (positive, negative, or 0), $j^bi^c = i^cj^b$ when b is even; $= i^{-c}j^b$ when b is odd.

From the formula, it follows that any member of the group can be expressed uniquely in the form $i^a j^b$, where a and b are integers (positive, negative, or 0).

4.5.2.2 The group just constructed is infinite, but the 3-element sets we shall examine below are of course finite, as are all things called "sets" in the present paper. We can make the group finite by imposing upon it the extra conditions: $i^4 = e$, $j^4 = e$; all exponents would then be computed modulo 4. Under the extra conditions, all of the arguments below will still go through as they stand.

Thus our example depends neither upon an assumption that our GIS is infinite, nor upon an assumption that it is finite.

The following lemma will be useful:

4.5.2.3 LEMMA: Within our group, POFI = {the powers of i} is a normal subgroup. That is:

(a) The group product of two powers-of-i is a power-of-i, and

(b) The group inverse of a power-of-i is a power-of-i, and

(c) For any i^a in the subgroup POFI, and any x in the entire group, the group-combination $xi^a x^{-1}$ is a power-of-i (member of POFI).

Proofs: (a) and (b) of the lemma are obviously true: $i^a i^c = i^{(a+c)}$; the group inverse of i^a is $i^{(-a)}$.

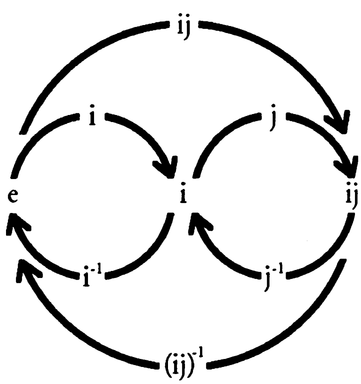
To prove (c) of the lemma, we express x as $i^c j^b$ for integers b and c. If the integer b is even, j^b commutes with all powers of i, and so x commutes with all powers of i (members of POFI); then $xi^a x^{-1} = i^a xx^{-1} = i^a$, which is a power-of-i (member of POFI). If the integer b is odd, then $xi^a x^{-1} = i^{(-a)} xx^{-1} = i^{(-a)}$, which is a power-of-i (member of POFI).

The lemma is proved.

4.5.2.4 We take the entire group under consideration to be both the objects and the intervals for a GIS, in the familiar way [of (4.1.2)]. Within that GIS, we consider the 3-element sets
A = {e, i, ij}, B = {e, j, ij}.

4.5.2.5 3-element sets A and B above have the same interval content. To see this, we compute the six non-identity intervals of A, which are i, i^{-1}, j, j^{-1}, ij, and $(ij)^{-1}$, according to the scheme of Figure 15 below.

Figure 15



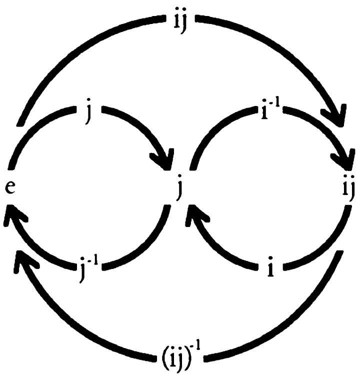
The figure is not exactly a transformational network in the sense of *GMIT.* The symbols labeling the arrows of the figure denote intervals, not transformations. That noted, we can see from the various arrow-labels that i is indeed the formal interval from e to i, defined as $e^{-1}i$, that j is indeed the formal interval from i to ij, defined as $i^{-1}(ij)$, and that ij is indeed the formal interval from e to ij, defined as $e^{-1}(ij)$. The label for the lowest arrow of the figure, running from right to left, announces that the interval from ij to e is $(ij)^{-1}$, the inverse of ij, which is to say the inverse of the interval-from-e-to-(ij). And so forth.

Now we compute the six non-identity intervals of B, which are also i, i^{-1}, j, j^{-1}, ij, and $(ij)^{-1}$, according to the scheme of Figure 16.

As on Figure 15, the arrow-labels on Figure 16 tell the story. The arrow from j to (ij) is labeled by the interval i^{-1}; that is because the interval from j to (ij), defined as $j^{-1}(ij)$, can be computed to be $j^{-1}(ij) = j^{-1}(ji^{-1}) = (j^{-1}j)i^{-1} = i^{-1}$. And so forth.

Thus (4.5.2.5) is established: A and B do indeed have the same interval content. (Each set also "contains" three intervals of e.)

Figure 16



4.5.2.6 B is not a transposition of A. The general GIS-transposition of the 3-element set A = {e, i, ij}, according to (4.1.3), is the set T(A) = {ex, ix, ijx}, where x is an element of the entire group.

Now T(A) = {x, ix, ijx}, and B = {e, j, ij}. If in fact B *were* the same unordered set as T(A), then the element x of T(A) would have to be some element of B; hence we would have either x = e (Case 1), or x = j (Case 2), or x = ij (Case 3). We shall show that each of the three Cases, in turn, cannot happen. That will prove (4.5.2.6).

Case 1: If x = e, then T(A) = {x,ix,ijx} = {e, i, ij}. This cannot be the same unordered set as B = {e, j, ij}, for the member j of the group is not the same as the member i of the group. *Case 1 cannot happen.*

Case 2: If x = j, then T(A) = {x,ix,ijx} = {j, ij, ij^2}. This cannot be the same unordered set as B = {j, ij, e}, since ij^2 is not e. *Case 2 cannot happen.*

Case 3: If x = ij, then T(A) = {x,ix,ijx} = {ij, i^2j, (ij)(ij)}. This cannot be the same unordered set as B = {e, j, ij}, since the member e of B is neither ij, nor i^2j, nor (ij)(ij), in our group. *Case 3 cannot happen.*

(4.5.2.6) is thus proved: 3-element set B is not a transposition of 3-element set A.

4.5.2.7 B is not an interval-preserving transformation of A. The general interval-preserving transformation of the 3-element set A = {e, i, ij}, according to (4.1.3), is the set P(A) = {xe, xi, xij}, where x is an element of the entire group.

Now P(A) = {x, xi, xij}, and B = {e, j, ij}. If in fact B *were* the same unordered set as P(A), then the element x of P(A) would have to be some element of B; hence we would have either x = e (Case 1), or x = j (Case 2), or x = ij (Case 3). We shall show that each of the three Cases, in turn, cannot happen. That will prove (4.5.2.7).

Case 1: If x = e, then P(A) = {x,xi,xij} = {e, i, ij}. This cannot be the same unordered set as B = {e, j, ij}, for the member j of the group is not the same as the member i of the group. *Case 1 cannot happen.*

Case 2: If x = j, then P(A) = {x,xi,xij} = {j, ji, jij}. This cannot be the same unordered set as B = {j, ij, e}, since e is a member of B and neither j, nor ji, nor jij is e. *Case 2 cannot happen.*

Case 3: If x = ij, then P(A) = {x,xi,xij} = {ij, iji, (ij)(ij)}. This cannot be the same unordered set as B = {e, j, ij}, since the member e of B is neither ij, nor iji, nor (ij)(ij), in our group. *Case 3 cannot happen.*

(4.5.2.7) is thus proved: 3-element set B is not an interval-preserving transformation of 3-element set A.

4.5.2.8 B is not a GIS-inversion of A. The general GIS-inversion of the 3-element set A = {e, i, ij}, according to (4.1.3), is the set I(A) = $\{xe^{-1}y, xi^{-1}y, x(ij)^{-1}y\}$, where x and y are elements of the entire group.

Now I(A) = $\{xy, xi^{-1}y, x(ij)^{-1}y\}$, and B = {e, j, ij}. If in fact B *were* the same unordered set as I(A), then the element xy of I(A) would have to be some element of B; hence we would have either xy = e (Case 1), or xy = j (Case 2), or xy = ij (Case 3). We shall show that each of the three Cases, in turn, cannot happen. That will prove (4.5.2.8).

Case 1: If xy = e, then y = x^{-1}j and I(A) = $\{xy, xi^{-1}y, x(ij)^{-1}y\}$ = $\{e, xi^{-1}x^{-1}, x(ij)^{-1}x^{-1}\}$, while B = {e, j, ij}. Since we are supposing that I(A) = B, it follows that the 2-element sets I(A') and B', formed by removing e from I(A) and B respectively, would have to be the same. We would have $I(A') = \{xi^{-1}x^{-1}, x(ij)^{-1}x^{-1}\}$ the same set as $B' = \{j, ij\}$. But the member $xi^{-1}x^{-1}$ of I(A') must be a power of i (Lemma 4.5.2.3), and neither member of set B' is a power of i.

So *Case 1 cannot happen.*

Case 2: If $xy = j$, then $y = x^{-1}j$ and $I(A) = \{xy, xi^{-1}y, x(ij)^{-1}y\} = \{j, xi^{-1}x^{-1}j, x(ij)^{-1}x^{-1}j\}$. If this were the same unordered set as $B = \{j, ij, e\}$, then the 2-element sets I(A') and B', formed by removing element j from sets I(A) and B respectively, would have to be the same. So we would have $\{xi^{-1}x^{-1}j, x(ij)^{-1}x^{-1}j\} = \{ij, e\}$. Now the member $xi^{-1}x^{-1}j$ of set I(A') is some-power-of-i times j (by Lemma 4.5.2.3). Since member e of set B' is not some-power-of-i times j, $xi^{-1}x^{-1}j$ must be the remaining member of set B', namely ij. Thus $xi^{-1}x^{-1}j = ij$. Canceling j from the right side of the equation, we infer that

(1) $xi^{-1}x^{-1} = i$.

There is only one remaining member of I(A') to match with only one remaining member of B'. We must have

(2) $x(ij)^{-1}x^{-1}j = e$.

Let us set $x = i^a j^b$. According to equation (1) just above, we have $i = xi^{-1}x^{-1} = (i^a j^b)(i^{-1})(j^{-b})(i^{-a})$. If the integer b were even, j^b would commute with everything in the group, and so i would $= i^{-1}$. This is not the case. So *b must be odd.*

Substituting $x = i^a j^b$ in equation (2) above, we have the equation $e = x(ij)^{-1}x^{-1}j = (i^a)(j^b)(ij)^{-1}(j^{-b})(i^{-a})j = (i^a)(j^b)j^{-1}i^{-1}(j^{-b})(i^{-a})j = (i^a)j^{(b-1)}(i^{-1})(j^{-b})(i^{-a})j$. Since the integer b is odd, the integer b–1 is even, $j^{(b-1)}$ commutes with (i^{-1}), and the expression at the end of the preceding sentence $= (i^a)(i^{-1})j^{(b-1)}(j^{-b})(i^{-a})j$, which is $(i^{(a-1)})(j^{-1})(i^{-a})j$, which—the j-exponents here being odd—is $(i^{(a-1)})(j^{-1})(i^{-a})j = (i^{(a-1)})(i^a)(j^{-1})j = i^{(2a-1)}$. In sum, we have inferred $e = i^{(2a-1)}$. But there is no integer a satisfying $2a-1 = 0$.[21]

[21]Here it is crucial that, when we considered modularizing our infinite group in (4.5.2.2), to make it finite, we imposed the extra condition $i^4 = e$ (rather than say $i^N = e$, where N is some odd number). If N were odd, then there would be no difficulty finding some number a satisfying $2a - 1 = 0$ *mod N*. But there is no such number a mod 4.

So *Case 2 cannot happen.*

Case 3: If xy = ij, then y = $x^{-1}ij$ and I(A) = {xy, $xi^{-1}y$, $x(ij)^{-1}y$} = {ij, $xi^{-1}x^{-1}ij$, $x(ij)^{-1}x^{-1}ij$}. If this were the same unordered set as B = {ij, j, e}, then the 2-element sets I(A') and B', formed by removing element ij from sets I(A) and B respectively, would have to be the same. So we would have {$xi^{-1}x^{-1}ij$, $x(ij)^{-1}x^{-1}ij$} = {j, e}. Now the member $xi^{-1}x^{-1}ij$ of set I(A') is some-power-of-i times j (by Lemma 4.5.2.3). Since member e of set B' is not some-power-of-i times j, $xi^{-1}x^{-1}j$ must be the remaining member of set B', namely j. Thus $xi^{-1}x^{-1}j$ = j. Canceling j from the right side of the equation, we infer that $xi^{-1}x^{-1}$ = e. But then $i^{-1} = x^{-1}ex$ = e. And that is not the case.

So *Case 3 cannot happen.*

(4.5.2.8) is thus proved: 3-element set B is not an GIS-inversion of 3-element set A.

Our demonstration is finished. 3-element sets A and B have the same GIS-interval content in our (non-commutative) GIS, but B is neither an interval-preserving transform, nor a GIS-transposition, nor a GIS-inversion of A.

In general, our argument will go through if we impose the conditions i^N = e and j^M = e, where M and N are both even integers, and N is bigger than 2. M must be even so that we can distinguish the odd powers of j from the even, modulo M. N must be bigger than 2, so that we can distinguish i from i^{-1} (which is $i^{(N-1)}$).

M, unlike N, may equal 2. In that case the group generated by i and j is "the dihedral group of order 2N." GIS structures possessing dihedral groups of intervals of order 2N, N being even and bigger than 2, have in fact already been used for music-analytic purposes in the standard literature. The author applied such a GIS—using the dihedral group of order 12—to the music of Schoenberg, in "Generalized Interval Systems for Babbitt's Lists, and for Schoenberg's String Trio," *Music Theory Spectrum* 17/1 (Spring 1995): 81–118. Edward Gollin applies such a GIS—using the dihedral group of order 8—to the music of Bartók, in "Some Unusual Transformations in Bartók's 'Minor Seconds, Major Sevenths'," to appear in the next issue of this journal.