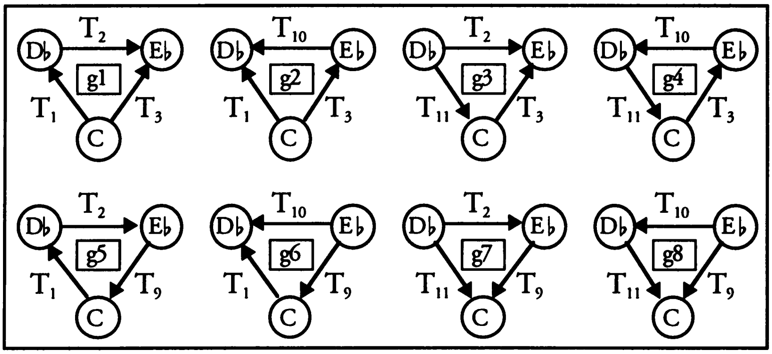
# Klumpenhouwer Networks, Isography, and the Molecular Metaphor


## Shaugn O'Donnell


David Lewin formally introduces Klumpenhouwer networks (henceforth K-nets) to music theorists in the article "Klumpenhouwer Networks and Some Isographies that Involve Them."[1] He presents K-nets as a variation of what I call Lewin networks (henceforth L-nets), which are networks of pitch classes related by transposition. L-nets articulate the dynamic nature of a set's internal structure by interpreting it as a web of transpositions rather than a static collection of pitch classes. This interpretation not only highlights a set's inherent ability to be a transformational model for subsequent musical gestures, but also allows numerous graphic possibilities for representing a single pitch-class set as shown in Example 1.

*Example 1. All eight L-nets modeling*[013].




One might consider each of these networks to be a specific interpretation of [013]'s interval-class vector, emphasizing

[1]Lewin, "Klumpenhouwer Networks and Some Isographies that Involve Them," *Music Theory Spectrum* 12/1 (1990): 83–120. Henry Klumpenhouwer first developed K-nets during his work on "A Generalized Model of Voice-Leading for Atonal Music" (Ph.D. dissertation, Harvard University, 1991).

particular transformations within the set.[2] In the most abstract sense, Klumpenhouwer expands Lewin's network model by allowing inversion operations within transformation networks. As Lewin suggests, this is "both simple and profound in its implications."[3] While transposition alone allows eight ways to model any given trichord, replacing one or more of the $T_n$ arrows with $I_n$ arrows allows nineteen additional—for a total of twenty-seven—possible interpretations. Example 2 again uses [013] to generate the nineteen new transformation networks. Klumpenhouwer's expansion is possible because one element of a dyad can map equally well onto the other via transposition or inversion. This simple abstraction results in the dramatic increase in graphic possibilities shown between these two examples. The practical applications of K-nets, however, are much more restricted than Example 2 suggests.

Lewin quickly points out that the primary advantage of K-nets is the ability "to interpret chords of different set classes with isographic networks."[4] To refresh the reader's memory: isographic networks are interpretations of sets yielding isomorphic graphs, or more simply, networks with the same configuration of nodes and arrows and sharing a mathematical correspondence between their analogous arrows. Furthermore, if the analogous transformation arrows are identical, networks are said to be strongly isographic.

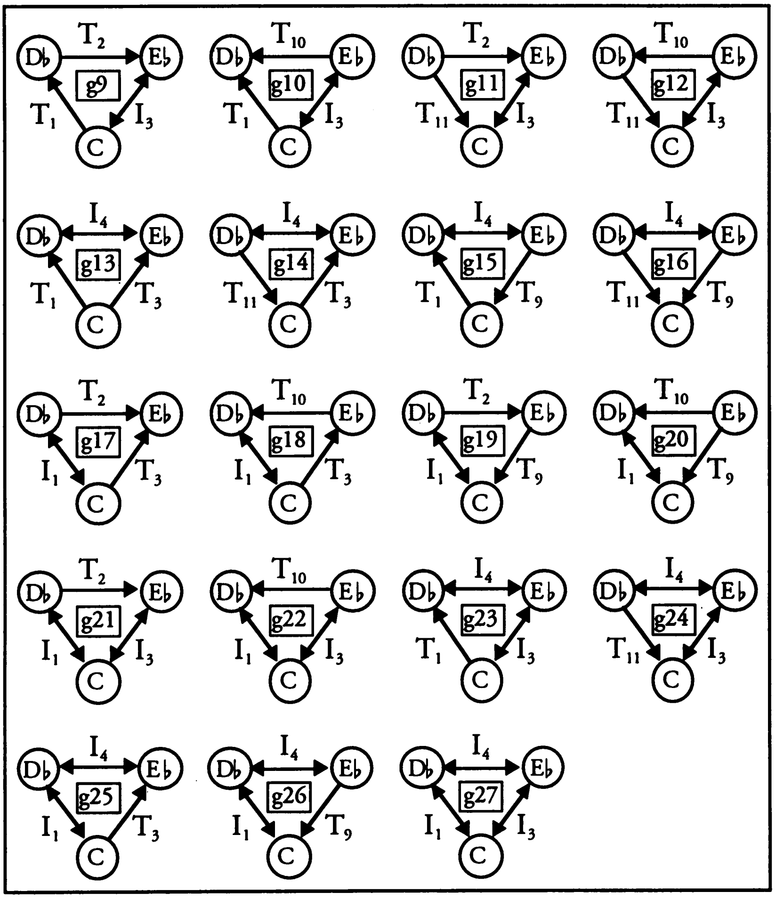
While Lewin uses isography to generate recursive supernetworks in his own analytical work, Klumpenhouwer uses this property to trace transformational mappings, which he interprets as voice leading. Transformational voices, particularly among chords belonging to different set classes, are my principal interest in this model, and the table in Example 3 compares the

[2]Digression #1: L-nets—as graphic models of the interval-class vector—are very useful in illustrating the structural differences between Z-related sets not evident from the vector itself. Howard Hanson implied this as early as 1960 when he called Z-related sets "isomeric twins," an earlier use of the molecular metaphor, in *Harmonic Materials of Modern Music: Resources of the Tempered Scale* (New York: Appleton-Century-Crofts, 1960), p. 22.

[3]Lewin, "Klumpenhouwer Networks," p. 84.

[4]*Ibid.*

*Example 2. Nineteen additional K-nets modeling [013].*



number of possible voice-leading mappings between sets of the same cardinality with the number of possible network models for sets of the given size.[5] Obviously, in terms of voice leading, it is excessive to have 32,768 different L-nets—not to mention having more than fourteen million K-nets—when there are only 720 different mappings between any two hexachords.

*Example 3. Comparison of possible mappings and networks.*

| Cardinality | Mappings | L-nets | "K-nets" |
|---|---|---|---|
| dyad | 2 | 2 | 3 |
| trichord | 6 | 8 | 27 |
| tetrachord | 24 | 64 | 729 |
| pentachord | 120 | 1,024 | 59,049 |
| hexachord | 720 | 32,768 | 14,348,907 |
| septachord | 5,040 | 2,097,152 | 10,460,353,203 |
| octachord | 40,320 | 268,435,456 | $3^{28}$ |
| nonachord | 362,880 | 68,719,476,736 | $3^{36}$ |
| decachord | 3,628,800 | $2^{45}$ | $3^{45}$ |
| monodecachord | 39,916,800 | $2^{55}$ | $3^{55}$ |
| dodecachord | 479,001,600 | $2^{66}$ | $3^{66}$ |

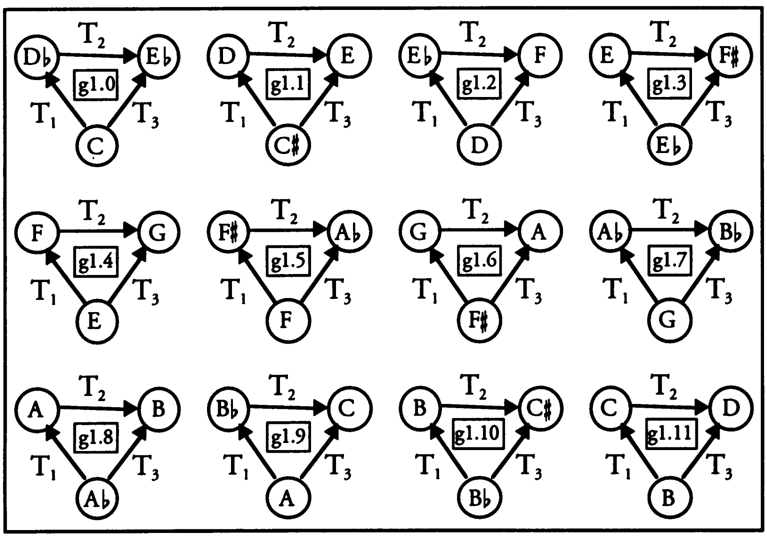
In practice, most of the additional interpretations elicited by Klumpenhouwer's theoretical abstraction do not afford any new relations among pitch-class sets, and the power of K-nets actually resides in a very small subset of these numerous possibilities. First, note that the [013] graphs in Examples 1 and 2 include three kinds of networks: there are those using only $T_n$ arrows (the L-

[5]The number of possible voice-leading mappings is calculated by n! where n equals the cardinality of the set. The number of possible L-nets is calculated by $2^p$ where p equals the number of interval classes contained in the set. The cardinality n can be used to determine the value of p with the equation $p = n(n - 1) \div 2$. For example: tetrachords contain $p = 4 \times 3 \div 2 = 6$ interval classes, therefore if $p = 6$, then $2^p = 2^6 = 64$ possible L-nets. The possible K-nets are calculated by a similar formula $3^p$. For example: if $p = 6$ (using tetrachords again) then $3^p = 3^6 = 729$ possible K-nets.

nets g1–g8), those using one $I_n$ and two $T_n$ arrows (g9–g20), those using one $T_n$ and two $I_n$ arrows (g21–g26), and one using all $I_n$ arrows (g27). I will immediately eliminate two of these groups—specifically g9–g20 and g27—as they are not well-formed, and by that I mean that their various transformational pathways are not equal. For example, traveling the direct route from C up to E♭ in g9 involves the path $I_3$, while the alternative route to E♭, with its layover at D♭, involves the paths $T_1$ and $T_2$, which sum to $T_3$ rather than $I_3$, and therefore (as $T_1 + T_2 \neq I_3$) the network is not well-formed.

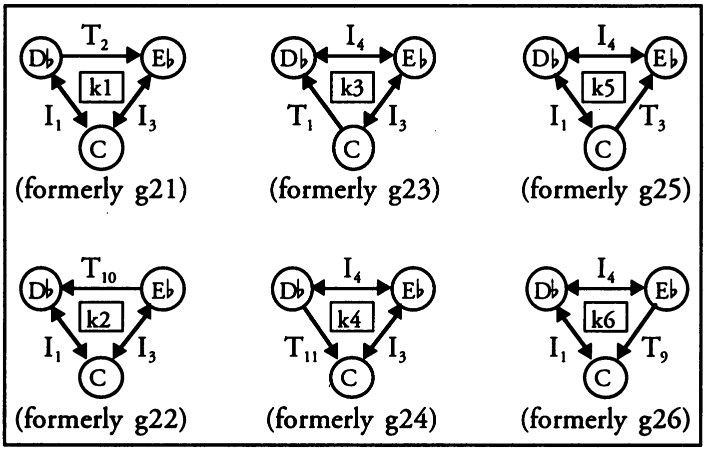
The first group, the L-nets, although well-formed, do not relate sets beyond traditional set-class boundaries. To exhibit strong isography two L-nets must model $T_n$-related pitch-class sets. Example 4 illustrates this phenomenon by building networks on each of the twelve pitch classes from the arbitrarily chosen g1, yielding the twelve members of $T_n$-type [013]. Any of the other seven L-nets in Example 1 similarly generate this complete $T_n$-type. In duplicating a subset of traditional set classes, specifically the $T_n$-type, these networks are redundant in terms of voice-leading transformations, and their primary value is their potential for replication on different structural levels.

*Example 4. Twelve strongly isographic L-nets modeling $T_n$-type* [013].



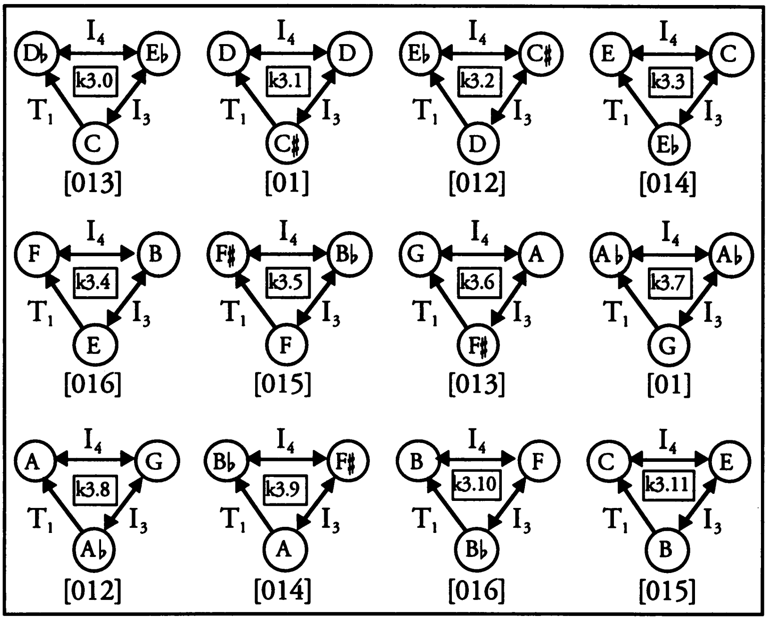
After excluding the eight L-nets and the thirteen ill-formed networks, only the six graphic interpretations shown in Example 5 remain viable. Arbitrarily taking the third of these graphs, k3, and building identical networks on each pitch class yields the twelve strongly isographic networks shown in Example 6. Significantly, these networks do not embody set-class relationships, rather, they form a family of similarly structured pitch-class sets *not* primarily related by transposition or inversion. The prime form of each set is given under its graph in the example. To differentiate these families from traditional set classes, I label such collections of strongly isographic networks Klumpenhouwer classes (henceforth K-classes). This is not Lewin's or Klumpenhouwer's terminology; they explore network isography in individual analytical cases, rather than as a generalizing force. Note that this class crosses traditional cardinality boundaries by incorporating two pitch-class dyads with doublings: {C♯, D} and {G, A♭}.[6] Each of the K-nets in Example 5 could generate similar K-classes.

*Example 5. The six well-formed K-nets modeling* [013].




[6]Digression #2: This property of K-classes intersects nicely with Robert Morris's KI complexes described in "K, Kh, and Beyond," *Music Theory in Concept and Practice* (Rochester: University of Rochester, 1997), pp. 275–306. Each connected pair of set classes in his abstract inclusion lattices has members belonging to the same K-class, and traveling more than one path is an interesting method of moving between different K-classes.

*Example 6. Sample K-class.*




Though not explicitly stated in Lewin's or Klumpenhouwer's work, the power of Klumpenhouwer's abstraction resides in the small subset of networks with the ability to generate classes such as those shown in the previous example. Referring back to Example 5, note that the six networks modeling [013] with this potential all incorporate two $I_n$ arrows and one $T_n$ arrow. In other words, each network contains a transpositional dyad and one other pitch class that is related by inversion to both members of the dyad.

Another way to think of this structure is as a partition of the set into two subsets linked by $I_n$ arrows. Note also that the vertically-aligned networks in the example are identical except for the reversal of the $T_n$ arrow directions. The left networks both isolate the inversional singleton C, the central networks isolate E♭, and the right isolate D♭, thereby exhausting all the partitions of the set other than the set itself. This demonstrates a property that is true for sets of any cardinality, and the number of practical K-nets arises from the total number of partitions less one. The table in Example 7 presents these more manageable numbers for sets of cardinalities three through six.[7] The reason for distinguishing practical from well-formed K-nets is that the direction of the $T_n$ arrows is not significant in terms of developing K-classes and voice leading, though it can suggest subtle interpretive differences on the level of individual sets and their higher-level replications.

*Example 7. Comparison of possible mappings and practical networks.*

| Cardinality | Mappings | Practical L-nets | Practical K-nets | Well-formed K-nets |
|---|---|---|---|---|
| trichord | 6 | 1 | 3 | 6 |
| tetrachord | 24 | 1 | 7 | 44 |
| pentachord | 120 | 1 | 15 | 480 |
| hexachord | 720 | 1 | 31 | 8,704 |

K-nets are thus part of a recent theoretical trend that parses pitch-class sets into subsets to explain relations among sets belonging to different set classes. Most notable in this movement are what I call singleton models, particularly Allen Forte's "unary transformations," David Lewin's "if-only," and Joseph Straus's "near-operations"—all manifestations of Forte's earlier $R_p$ relation.[8] The sizable body of musical works discussed by Forte,

[7]The formula $2^{n-1} - 1$, where n equals the cardinality of the set, generates the number of possible practical K-nets, while flipping each of the $T_n$ arrows generates the number of well-formed K-nets. The excluded unity/null partition corresponds to practical L-nets as implied by Example 4.

[8]To refresh the reader's memory: pitch-class sets of cardinality n are $R_p$ related if they share a subset of the cardinality n – 1. Forte introduces unary transformations in "New Modes of Linear Analysis," paper presented at the Oxford University Conference on Music Analysis (1988), and the Rp relation in *The Structure of Atonal Music* (New Haven: University of Yale Press, 1973) p. 46*ff*; Lewin introduces if-only in "Transformational Techniques in Atonal and other Music Theories," *Perspectives of New Music* 21 (1982–83), pp. 312–371,

Lewin, Straus, and Klumpenhouwer in introducing their respective models suggests that partitioning is a fruitful analytical avenue. Furthermore, the number of theorists independently arriving at and exploring similar approaches, as well as the increasing number of current dissertations and theoretical papers on related topics, suggests that something about these subset-oriented models resonates well with us as musicians.[9]

The development of all these new transformations is a reaction to the exclusive nature of the traditional transposition and inversion operators, which allow at most twenty-four relations among the 4,096 possible pitch-class sets. The exclusivity of traditional set classes not only leads towards a fragmented analytical approach that traces independent paths for each prominent set class within a given musical work, but often also leaves large temporal gaps between two "adjacent" members of the same set class.

At the opposite end of the spectrum one finds John Roeder's voice-leading model which traces independent lines for each registral voice in a passage, thereby allowing any set to transform into any other.[10] Many theorists' work in this area is a search for

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and further develops the concept in *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987); and Straus introduces near-operations in "Voice Leading in Atonal Music," *Music Theory in Concept and Practice* (Rochester: University of Rochester Press, 1997), pp. 237–274.

[9]A sampling of recent items—within the past few years—includes: Brian Alegant, "When Even Becomes Odd: A Partitional Approach to Inversion," paper presented at the Society for Music Theory Annual Conference (Baton Rouge, 1996); the transformational voice leading component of Michael Buchler's "Setmaker" software for Macintosh computers (version 5.0, ©1996); Edward Jurkowski, "A Model for Non-Equivalent Set-Class Associations: The Interval-Difference Network," paper presented at the Music Theory Society of New York State Annual Conference (Stony Brook, 1996); Stephen Nuss, "Tradition and Innovation: The Art Music of Post-War Japan" (Ph.D. dissertation, City University of New York, 1996); and, Ian Quinn, "Fuzzy Transposition of Pitch Sets," paper presented at the Society for Music Theory Annual Conference (Baton Rouge, 1996).

[10]Roeder, "A Theory for Voice Leading in Atonal Music" (Ph.D. dissertation, Yale University, 1984), and "Voice Leading as Transformation," *Musical Transformations and Musical Intuition: Eleven Essays in Honor of David*

a satisfactory location along this exclusivity-promiscuity continuum; for example, consider the following chain of events in Forte's work: (1) he codifies set classes based on transposition and inversion; (2) he introduces the $R_p$ relation to allow relations beyond the limits of the above set classes; (3) he complains that the $R_p$ relation's practical value is limited because it relates too many sets; and (4) he develops unary transformations as a refinement of the $R_p$ relation. While traditional operations are overly exclusive and Roeder's abstract model is overly promiscuous, my ideal location on the continuum currently resides in two transformations called dual transposition and dual inversion.[11]

Dual transposition is a transformation using two simultaneous transpositions, that is, some voices move at $T_n$ while the remainder move at $T_{n+x}$, resulting in the compound transformation $T_n/T_{n+x}$.[12] More formally, imagine two pitch-class sets J and K. Partition each of these sets into two discrete subsets encompassing the entire set, so that they become the partially ordered sets J = <{j1}, {j2}> and K = <{k1}, {k2}>. The cardinalities of these sets and subsets are irrelevant, except that

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*Lewin*, ed. Raphael Atlas and Michael Cherlin (Roxbury, MA: Ovenbird Press, 1994), pp. 41–58. Note that while Roeder's model has the ability to relate all sets, he does not in fact use it to do so, instead he applies other transformational criteria as filters in his analyses.

[11]Previous papers by the author on these transformations are: "Harmonic Progression and Voice Leading in the First of Stravinsky's *Movements for Piano and Orchestra*," paper presented at the Music Theory Society of New York State Annual Conference (Flushing, 1993); "Transformational Voice Leading in Two Songs by Charles Ives," paper presented at the Society for Music Theory Annual Conference (Baton Rouge, 1996); and, "Transformational Voice Leading in Atonal Music" (Ph.D. dissertation, City University of New York, 1997). An earlier version of this article was presented at the University of Iowa (April 1998), and I would like to thank those present, particularly Michael Buchler and Nancy Rogers, for their insightful comments and questions. Stephen Soderberg uses "dual inversions" toward different ends in "Z-Related Sets as Dual Inversions," *Journal of Music Theory* 39/1 (1995): 77–100.

[12]My notation includes a slash between the two operators to distinguish dual transformations from composite operations, which operate sequentially on the same object rather than simultaneously on different objects.

the cardinality of each subset must be at least one. If $T_n$ transforms {j1} into {k1}, and $T_{n+x}$ transforms {j2} into {k2}, then $T_n/T_{n+x}$ transforms J into K. Dual inversions work identically except that $I_n$ transforms {j1} into {k1} and $I_{n+x}$ transforms {j2} into {k2}, and therefore $I_n/I_{n+x}$ transforms J into K. In other words, some voices move at $I_n$ while the remainder move at $I_{n+x}$. These transformations are best illustrated by musical examples.

Example 8 shows a progression of four chords labeled c1–c4. The middle of the progression, c2 → c3, comprises a member of set-class [0237] transforming into a member of set-class [0257]. By definition, traditional transposition or inversion cannot generate this transformation, but as notated at the bottom of the example below the mappings, the dual transposition $T_1/T_{10}$ can. That is, $T_1$ transforms the bass-clef dyad, while $T_{10}$ transforms the treble-clef dyad. In this and subsequent examples, solid and dashed arrows distinguish between the two $T_n$ levels in the mappings. Intellectually this process involves first partitioning the c2 tetrachord {G, B♭, E♭, A♭} into two discrete dyads {G, B♭} and {E♭, A♭}, and then tracing the dyads' independent transpositional paths into the dyadic subsets {A♭, C♭} and {D♭, G♭} of the c3 tetrachord {A♭, C♭, D♭, G♭}. Put in simple terms, I am asking the reader to hear the [03] dyad move up a half step and the [05] dyad move down a whole step.

The progression c2 → c3 illustrates the power of dual transpositions to generate mappings among different set classes, but they can also generate mappings among members of the same class, sometimes in ways that are more satisfactory than traditional operations. I could interpret the initial progression c1 → c2 as a traditional $I_8$, but that would involve hearing the dyads flip as shown by the mappings in Example 9. Such an interpretation is plausible for the bass-clef dyad, as B♭ acts as a common tone, but it is less satisfactory for the treble-clef dyad. On the other hand, the dual transposition $T_9/T_3$ shown in Example 8 suggests hearing the symmetry of the progression as the two dyads move away from each other by complementary $T_n$ values. It is much easier to hear the motion of the parallel intervals than the pitch-class inversion in this particular case.

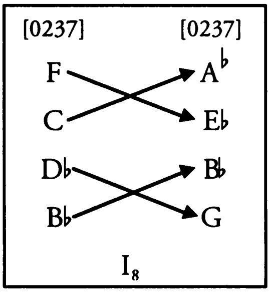
*Example 8. Dual Transposition.*

| c1 | c2 | c3 | c4 |
|---|---|---|---|
| [0237] | [0237] | [0257] | [0257] |
| F ------→ | A♭ ------→ | G♭ ------→ | F |
| C ------→ | E♭ ------→ | D♭ ------→ | C |
| D♭ ——→ | B♭ ——→ | C♭ ——→ | B♭ |
| B♭ ——→ | G ——→ | A♭ ——→ | G |

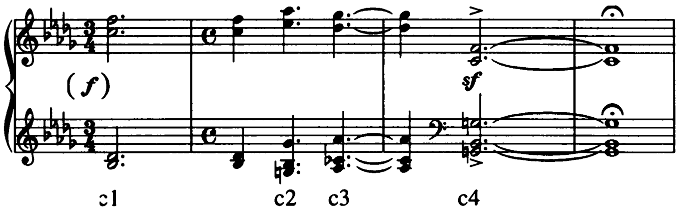
$T_9/T_3$ $T_1/T_{10}$ $T_{11}/T_{11}$

The final progression c3 → c4 illustrates a situation in which traditional and dual transpositions agree. All the pitches of c3 move down a half step to generate c4, but owing to the emphasis on the independent dyads in the three previous chords, the dyadic partition remains very audible as reflected by the notation $T_{11}/T_{11}$. The source of this abstract chord progression is the opening of Aaron Copland's *Piano Sonata*, shown in Example 10. I normalized the passage in the earlier example for reasons of clarity by removing the octave displacement in the transformation c3 → c4, by omitting the orchestrational doubling in the left hand in chords c2–c4, and by eliminating any metric reference.

*Example 9. Inversional mapping for $c_1 \rightarrow c_2$.*




*Example 10. Aaron Copland,* Piano Sonata, *I, mm. 2–5.*




The Bartók excerpt shown in Example 11 illustrates dual inversions. The chord progression c1–c4 contains members of two different set classes: [0145] and [012]. The transformations among the last four chords reconfirm the power of dual transformations to generate mappings among sets from different classes, and further demonstrate that they can also span traditional cardinality boundaries. It is important to note that these sets have an equal number of pitches, despite the difference in pitch-class cardinality. In this model it is therefore possible to incorporate pitch-class doublings as components of independent voices, allowing analysts to distinguish between orchestrational doublings as in the previous Copland example and occasional pitch-class duplications between different voices as in the Bartók example. The progression c1 → c2 shows that dual inversion, like dual transposition, can sometimes generate more satisfactory mappings than traditional operations between members of the same set class. Example 12 presents the two possible operational mappings between c1 → c2 for comparison. As the title of the piece—"Minor Seconds, Major Sevenths"—suggests, it is the inversional transformation of the dyads that is most significant in this excerpt; the tetrachords result from this process, rather than generating it.

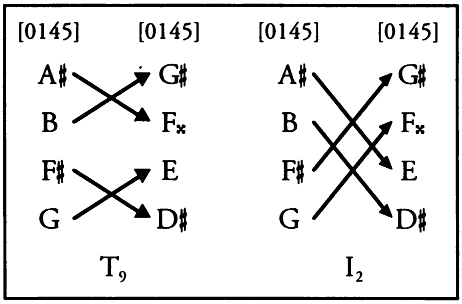
*Example 11. Dual inversion; Béla Bartók, No. 144,* Mikrokosmos, *VI, m. 39.*

| c1 | c2 | c3 | c2 | c4 |
|---|---|---|---|---|
| [0145] | [0145] | [012] | [0145] | [012] |

A♯ - - - → G♯ - - - → F♯ - - - → G♯ - - - → A
B - - - → F𝄪 - - - → G - - - → F𝄪 - - - → B♭
F♯ ——→ E ——→ G ——→ E ——→ G♯
G ——→ D♯ ——→ A♭ ——→ D♯ ——→ A

$I_{10}/I_6$ $I_{11}/I_2$ $I_{11}/I_2$ $I_0/I_5$

*Example 12. Traditional mappings for $c_1 \rightarrow c_2$.*




Dual transformations and K-nets are two sides of the same coin. Example 13 interprets the Copland progression as four K-nets, k1–k4, representing the corresponding chords c1–c4. All four K-nets have the same configuration of nodes and arrows, identical $T_n$ arrows, and analogous $I_n$ arrows that differ by a fixed amount. Lewin defines this relation as positive isography (strong isography is the subset in which $I_n$ arrows differ by 0), represented by $\langle T_n \rangle$, where n equals the difference between analogous $I_n$ arrows.[13] In this progression $\langle T_0 \rangle$ transforms k1 into k2, meaning the two K-nets are strongly isographic, that is, members of the same K-class. The transformations $\langle T_{11} \rangle$ and $\langle T_{10} \rangle$ follow, transforming k2 into the positively isographic networks k3 and k4. Example 13 notates the series of transformations $\langle\langle T_0 \rangle, \langle T_{11} \rangle, \langle T_{10} \rangle\rangle$ under the mappings (identical to those in Example 8 above) suggested by analogous nodes. $\langle T_n \rangle$ transformations group K-classes into families of twelve classes—one for each possible value of n (0–11)—for a total of 144 related networks. Example 14 interprets the four chords, c1–c4, of the Bartók progression as K-nets k1–k4. Note that these networks have the

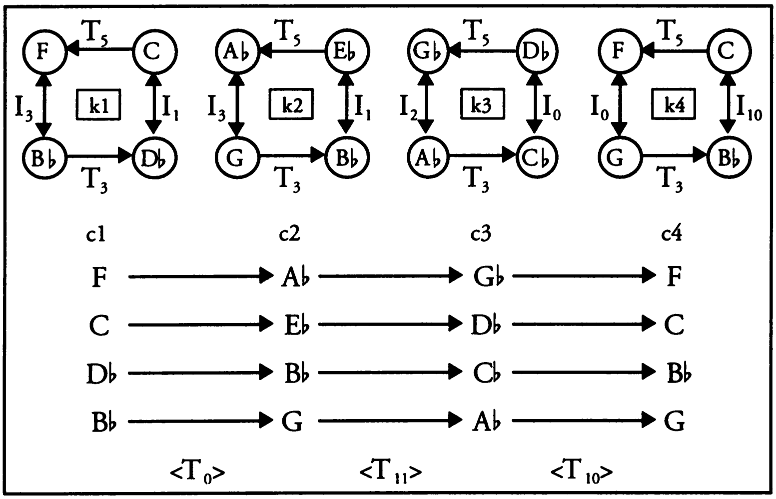
[13]Lewin formally defines positive and negative isography in "Klumpenhouwer Networks." He further develops the relation in "A Tutorial on Klumpenhouwer Networks, Using the Chorale in Schoenberg's Opus 11, No. 2," *Journal of Music Theory* 38/1 (1994): 79–101, as does Klumpenhouwer in "A Generalized Model."

same configuration of nodes and arrows, complementary $T_n$ arrows, and that their analogous $I_n$ arrows sum to the same value.[14] Lewin defines this relation as negative isography, represented by $<I_n>$, where n equals the sum of the analogous $I_n$ arrows. Example 14 notates the series of transformations $<<I_4>, <I_1>, <I_1>, <I_5>>$ below the mappings (identical to those in Example 11 above). The concept of negative isography further expands the family of K-classes to 288 related networks. K-nets derive their analytical power from these extended families of positively and negatively isographic networks.[15]

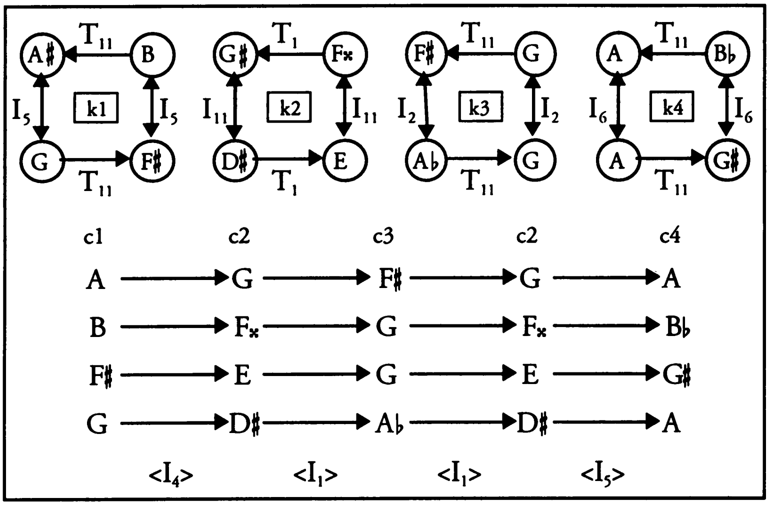
[14]It is merely coincidence that the analogous $I_n$ arrows also differ by a fixed amount in this example. This property arises from the identical $I_n$ arrows within each network, which, in turn, are simply a result of configuring these particular nodes this particular way.

[15]Digression #3: George Perle pioneered the research into these relations decades earlier in much of his theoretical, analytical, and compositional work. Of particular relevance is his work on P/I dyads and sum and difference scales in *Twelve-Tone Tonality*, 2nd ed. (Berkeley: University of California Press, 1996). Perle specifically points out some of the connections between Klumpenhouwer's work and his own in a letter to *Music Theory Spectrum* 15/2 (1993): "A Klumpenhouwer network is a chord analyzed in terms of its dyadic sums and differences... The analysis of tetrachords into their component dyadic sums and differences as a means of defining the relations between different tetrachords is demonstrated in the third chapter of *Twelve-Tone Tonality*." [pp. 300, 302] While a detailed comparison of K-nets and Perle's work is beyond the scope of this study, there are two fundamental differences worth noting here. First, the two models have different musical orientations. Perle's is primarily a compositional model or perhaps even a precompositional space, while Klumpenhouwer's is a graphic analytical model. Second, and more importantly in the context of this article, specific mappings are an essential part of Klumpenhouwer's approach, while Perle's cyclical sets imply no particular voice leading. The new edition of *Twelve-Tone Tonality* does include a brief chapter on voice leading, "Voice Leading Implications of Sum Tetrachords" (pp. 177–182), but Perle seems to equate voices with registral lines and offers very few musical realizations of the progressions he discusses.

*Example 13. K-nets modeling the Copland progression.*




*Example 14. K-nets modeling the Bartók progression.*



The previous two examples demonstrate that one could notate any $<T_n>$ transform as $T_n/T_{n+x}$, and any $<I_n>$ transform as $I_n/I_{n+x}$. That is, positively isographic K-nets can model any dual transposition and negatively isographic K-nets can model any dual inversion. A valid question at this point might be: if dual transformations are merely another way of describing K-nets, why introduce a new model? The answer is that while the two models have substantial similarities, they are complementary rather than identical. Returning to the Copland progression once again, the individuality of the two approaches is perhaps most obvious in the context of listener perception. Hearing the progression c1 → c2 as $T_3/T_9$ requires listeners to focus on the right hand moving up three semitones, while the left hand moves down three. In contrast, hearing the progression as $<T_0>$ requires listeners to focus on the hands wedging apart symmetrically. Most significantly, the precise linear transformation is irrelevant; the only requirement is that the two hands travel the same distance. In fact, c1 → c1 is as much $<T_0>$ as c1 → c2.

Continuing on to the progression c2 → c3 (which—unlike c1 and c2—does not comprise members of the same K-class) further highlights the differences between dual transposition and $<T_n>$. As I mentioned above, interpreting c2 → c3 as $T_{10}/T_1$ involves hearing the right hand move down a whole step and the left hand move up a half step. $<T_{11}>$ requires the listener to hear the asymmetry or skewing of the wedge, specifically that one of the hands is a semitone "off." The final motion, c3 → c4, makes the difference even more prominent as the two hands move in similar motion. Hearing $T_{11}$ as a wedge skewed by a whole step, $<T_{10}>$, is quite counterintuitive. The fundamental difference between the two processes in question is one of internal versus external orientation, perhaps equivalent to the more historically familiar "vertical" versus "horizontal." Dual transposition (like traditional transposition) transforms pitch classes through a specified external or horizontal motion, and similarly, dual inversion transforms pitch classes by flipping them externally or horizontally around a given axis. K-nets' internal transpositions and inversions transform pitch classes by a vertical distance, or through a vertical axis. This process makes the set dynamic, yet renders the transformation static; the horizontal voice leading results from equivalent locations in the internal or vertical structure rather than from any particular linear transformation. The wedging process does imply a specific kind of contrapuntal linear motion (complementary contrary motion), but the fact that the distance traveled along the wedge is irrelevant to the network process underscores its internal emphasis. Furthermore, $<T_0>$ wedging is relatively easy to hear in pitch space, but becomes more difficult as the value of $<n>$ increases, and even more so as the musical universe shifts to pitch-class space.

Comparing Examples 8 and 13, note the relation between corresponding $<T_n>$ and $T_n/T_{n+x}$ transformations: the $<T_n>$ subscript equals the sum of the dual subscripts, or $<n> = 2n + x$. The same holds true for $<I_n>$ and $I_n/I_{n+x}$ (compare Examples 11 and 14). Example 15 illustrates the significance of this difference in notation. The example shows three different $<T_0>$ transformations from the first chord of the Copland excerpt; the first, c1 → c2.0, being identical to the original. The next two, however, result in chords c2.1 and c2.2, different progressions than those occurring in Copland's actual music. In fact, there are nine other possible $<T_0>$ transformations yielding different second pitch-class sets, as $<T_0>$ represents strong isography, or members of the same K-class. The dual transpositions, on the other hand, distinguish among these twelve different progressions, and therefore provide a more successful foreground model of the voice leading. It is the x-value—the difference between the two transposition subscripts—and its effect on set-class metamorphosis that helps characterize the distinct transformations. Note that each of the $T_n/T_{n+x}$ transformations in this example has a different x-value—6, 8, and 2—which results in c2.0, c2.1, and c2.2 being members of different set classes.[16]

[16]It is possible to have different x-values result in members of the same set class, but dual transformations with the same x-value will—assuming the same partition—always result in the same set-class metamorphosis. Just as $T_5/T_7$ transforms c1 [0237] into c2.2 [0147] in Example 15, any x = 2 dual transposition of c1 ($T_0/T_2$, $T_1/T_3$, etc.) results in a member of [0147]. For more

*Example 15.* $<T_n>$ *versus* $T_n/T_{n+x}$

| c1 | c2.0 | c1 | c2.1 | c1 | c2.2 |
|---|---|---|---|---|---|
| [0237] | [0237] | [0237] | [025] | [0237] | [0147] |
| F - - → A♭ | | F - - → A | | F - - → C | |
| C - - → E♭ | | C - - → E | | C - - → G | |
| D♭ ——→ B♭ | | D♭ ——→ A | | D♭ ——→ G♭ | |
| B♭ ——→ G | | B♭ ——→ G♭ | | B♭ ——→ E♭ | |
| $<T_0>$ | | $<T_0>$ | | $<T_0>$ | |
| $T_9/T_3$ | | $T_8/T_4$ | | $T_5/T_7$ | |

If dual transformations generally offer a better foreground voice-leading model, as I suggested above, one might ask what is the advantage of continuing to use K-nets? There are at least two extremely significant reasons not to displace K-nets with dual transformations. The first is the extraordinary potential for network recursion. Dual transformations do not particularly lend themselves to replication on different structural levels, while K-nets provide an elegant recursive model. The second reason is more subtle. Recall that one weakness in deriving voices from K-nets is the model's internal orientation, that is, the mappings result from equivalent nodes rather than any specific linear transformation. Dual transformations remedy this problem by describing the same voice mappings using two simultaneous linear operators; in other words, they render dynamic the between-set transformation. Although a successful voice-leading model should have an external or horizontal orientation, the internal dynamism of the network model captures an invaluable dimension of musical structure that I think of as set integrity, or

information on x-values see chapter 1.4 of the author's "Transformational Voice Leading in Atonal Music."

the modern equivalent of chord quality in tonal music. With its horizontal emphasis and explicit partitioning of sets, dual transformations do not satisfactorily model the cohesive quality of individual chords or sets, particularly in homorhythmic contexts. Although K-nets implicitly embody the same partitions, the graphic approach more successfully captures the unity of individual sets via a molecular metaphor.

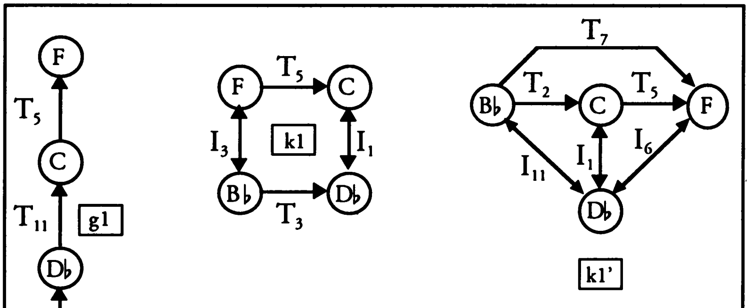
One of the most compelling aspects of this metaphor is the correspondence between different graphic interpretations of a set and the concept of isomers.[17] Example 16 includes three of the eight possible network interpretations of Copland's first chord, c1. Remember that there are 108 well-formed networks (64 L-nets and 44 K-nets) for every tetrachord, but only eight practical networks (seven K-nets and a single L-net). The first network, g1, is an L-net (arbitrarily arranged in ascending registral order), while k1 is a 2+2 K-net and k1' is a 3+1 K-net. The example illustrates that networks are metaphoric structural formulas, set-class labels are molecular formulas, and that making the K-net partitioning explicit generates a rough equivalent to condensed structural formulas.[18] These condensed structural formulas are a little misleading, and only provide sufficient information about a set in conjunction with the molecular formula. That is, using k1 as an example, [03][05] represents a complete K-net family, and not just "iso"[0237]. It would be more precise to use [03][27] as the condensed structural formula for k1, but the more abstract [03][05] better facilitates pivoting between set classes and K-net families for analytical or compositional purposes. The complete

[17]Jeffrey Johnson also points out this correspondence in *Graph Theoretical Models of Abstract Musical Transformations: An Introduction and Compendium for Composers and Theorists* (Westport, CT: Greenwood Press, 1997): 28–29.

[18]Digression #4: Morris's rigorous work on set-class unions in "Pitch-Class Complementation and its Generalization," *Journal of Music Theory* 34/2 (1990): 175–245, formalizes many of the properties of my condensed structural formulas without K-nets in mind. Significantly, while Morris generalizes non-intersecting unions, K-nets incorporate the possibility of pitch-class duplication. Such pitch-class "weighting"—which results in the cardinality differences in some K-classes—corresponds metaphorically to molecular isotopes.

list of [0237] isomers is: [013][0], [016][0], [027][0], [037][0], [01][05], [02][04], and [03][05]. Each of the preceding isomers represents a K-net family, and the ability to reinterpret pitch-class sets this way has substantial voice-leading ramifications.

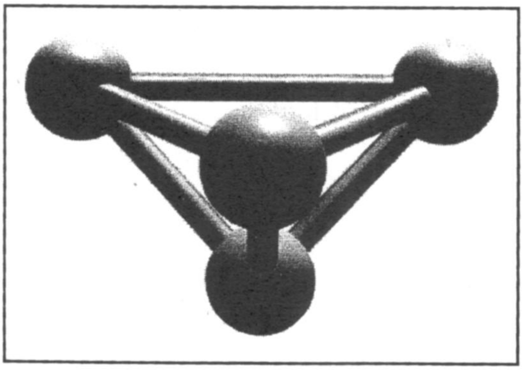
*Example 16. Isomers.*




| | | | |
|---|---|---|---|
| structural formula: | gl | kl | kl' |
| molecular formula: | [0237] | [0237] | [0237] |
| condensed structural formula: | [0237] | [03][05] | [027][0] |

Taking the metaphor a little further, I visualize K-nets as three-dimensional ball-and-stick models with nodes standing in for atoms, and transformations functioning as bonds. Example 17 illustrates the tetrahedral structure implied by kl'. This molecular model best captures the way I understand chord or set quality as a dynamic, yet cohesive, network of individual elements. Thinking of transformations as nondirectional bonds, rather than arrows, corresponds to my notion of practical, as opposed to well-formed, networks. I urge the reader to avoid letting the rigid appearance of the "sticks" render the model static, but rather to keep in mind the highly dynamic atomic processes they typically represent, such as sharing or stealing electrons. Unfortunately, the incompatibility of a three-dimensional model and this two-dimensional medium forces my K-nets into the "flat" form shown in my other examples.

*Example 17. Three-dimensional "ball-and-stick" network.*



Obviously the molecular metaphor has many limitations; for example, in addition to the configuration of nodes and arrows, the specific content of each node is essential to chemistry, while I am assuming the transpositional equivalence of pitch classes. Imagine a chemical universe in which lithium sulfide ($Li_2S$) is just a semitone up from water ($H_2O$)! More significantly, concrete physical properties limit the bonding potential of elements in molecular nodes, while pitch-class nodes have no such limitations and may bond with any other. The lack of natural limitations on pitch-class bonds in L-nets and K-nets raises questions about the configuration of the arrows in these networks. Compare the isomers k1 and k1' in Example 16. They model identical tetrachords, but k1 has four arrows and k1' has six; Lewin uses these configurations for 2+2 (k1) and 3+1 (k1') K-nets. In graph terminology k1 is a 2-regular graph, while k1' is a complete graph.[19] In 2-regular graphs each node connects to two others, thereby creating a single cycle, in this case a square because there are four nodes. Complete graphs differ in that each node connects to all the others, leading to my tetrahedron in Example 17. If networks must be well-formed, then most of these arrows

[19]Johnson discusses K-regular and complete graphs in Part III of *Graph Theoretical Models*, 51–61. This terminology assumes that the transformational arrows are unrestricted, that is, I am still thinking of $T_n$ arrows as being nondirectional for all practical purposes.

are redundant, and if each node has at least one arrow (see g1 in Example 16), then the remaining arrows are all implied. I interpret all L-nets and K-nets as complete graphs forming solid shapes in three dimensions (g1 and k1 are also tetrahedrons), but prefer notating them as 2-regular graphs for reasons of clarity when working in two dimensions.

In an effort to bring together several of the above threads I will conclude with a brief analysis of an excerpt from Schoenberg's Op. 11, No. 2; a short progression from the same "chorale" passage Lewin uses for his K-net tutorial. Example 18 includes the music and relevant K-nets from Lewin's more extensive analysis.[20] The K-nets, k1–k4, at the bottom of the example interpret the chords, c1–c4, at the top of the example. Note that there are two pairs of positively isographic networks, k1 → k2 and k3 → k4, and that these pairs are negatively isographic to each other. The two-plus-two pairing of positively and negatively isographic networks suggests that the graphs themselves may follow a transformational path similar to the pitch classes within the individual networks. The supernetwork K illustrates the isographic relation between the network-of-networks and k1–k4. As Lewin suggests: the supernetwork "thus interprets the *progression* of chords 1–2–3–4 by a K-structure that exactly reproduces, on a higher level, the structure that interpreted each chord."[21] Lewin finds great significance in such recursive structures, and suggests a potential hierarchical voice-leading model by his choice of terminology:

> When a lower-level Klumpenhouwer Network is interpreting a chord, and a higher-level network-of-Networks is interpreting a progression of chords..., I noted that one could conceive of the higher-level network as "prolonging" the given progression. This potentiality of the system, observed again and again in the article, can afford an especially compelling rationale for asserting one

[20]The K-nets in Example 18 are drawn from Lewin's examples 10, 12 and 13 in "A Tutorial." I have modified his notation to match my own.

[21]Lewin, "A Tutorial," 91 (emphasis original). Lewin's more in-depth analysis of this work takes things one step further, generating an even higher level network using $<<T_n>>$ and $<<I_n>>$ to model the overall progression of progressions. Theoretically these nestings could go on indefinitely.

particular Klumpenhouwer Network rather than another to interpret a given chord. I found it suggestively comparable, methodologically, to the ways in which a choice among foreground readings in a Schenkerian analysis can be influenced by middleground considerations.[22]

Unfortunately, the analytical elegance of network recursion loses some of its charm if one examines exactly what it implies on the musical surface. The first progression, c1 → c2, illustrates this point. Lewin, unlike Klumpenhouwer, does not explicitly demonstrate the voice leading implied by his K-nets. The mappings labeled "Recursive Structure" in Example 18 illustrate the voices generated by the supernetwork K, while the mappings labeled "Adjacencies" are a series of dual transpositions that model my interpretation of the musical surface.

A comparison of the two sets of mappings for the progression c1 → c2 highlights the abstract nature of the recursive voice leading. The progression consists of a single melodic motion in the highest register heard against a sustained three-voice chord. The $T_0/T_2$ transformation of my foreground interpretation successfully captures this singleton partition, as well as the non-motion of the lower three voices. The $\langle T_7 \rangle$ mappings, on the other hand, retain the alto and bass voices, but suggest a soprano-tenor exchange and a 2+2 partition. Hearing the soprano A4 map into an already sounding E4, and the E4 map into B4 in the middle of its duration, strongly contradicts my experience of the progression.

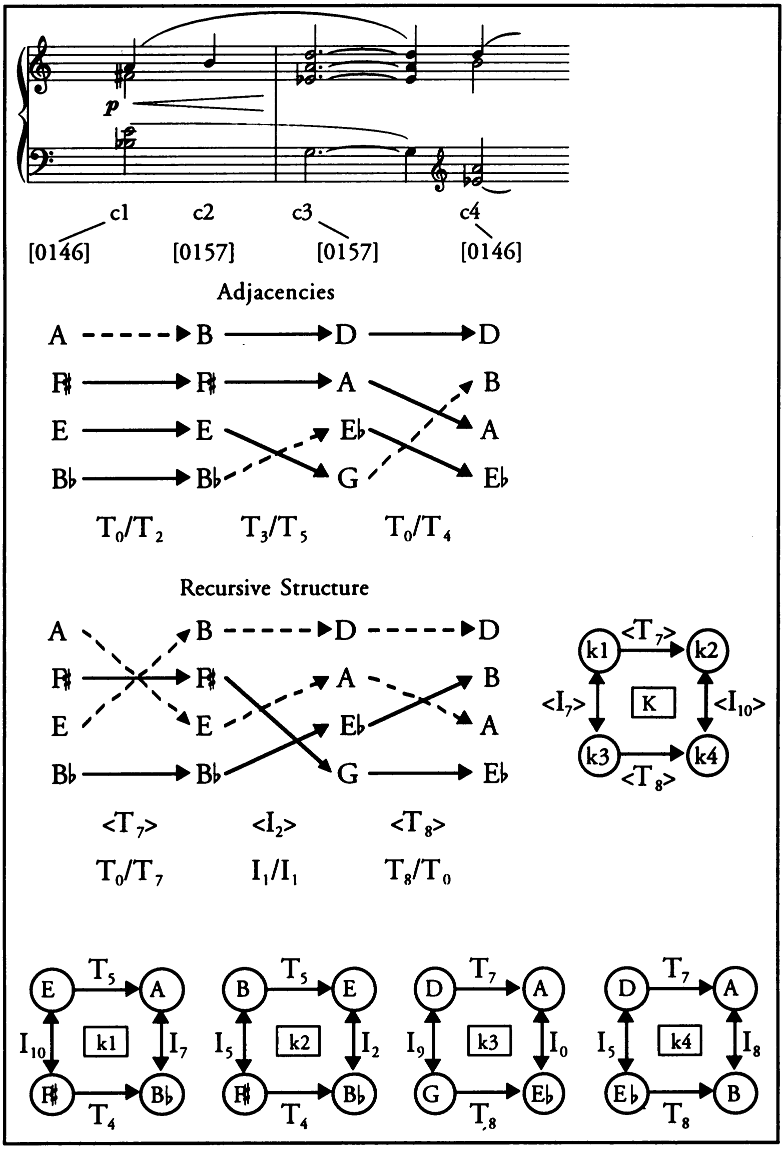
[22]Lewin, "Klumpenhouwer Networks," 115. Digression #5: The importance of supernetworks in Lewin's work raises the question: how meaningful an isomorphism is network recursion? In the most abstract sense, both $T_n$ and $\langle T_n \rangle$ are about differences, while $I_n$ and $\langle I_n \rangle$ are about sums. Despite this mathematical similarity, keep in mind that traditional and bracketed transformations operate on very different objects. $\langle T_n \rangle$ and $\langle I_n \rangle$ transform Klumpenhouwer arrows among isographic networks, while $T_n$ and $I_n$ transform pitch classes within a single network. This suggests a substantial functional difference (that is, transformations of transformations versus transformations of pitch classes), but within the context of the graphic model they both transform nodes. Currently I find the isomorphism musically meaningful and useful on a middleground level, and somewhat less convincing on a background level incorporating supernetworks invoking $\langle\langle T_n \rangle\rangle$ or $\langle\langle I_n \rangle\rangle$.

The common tones in the motion c3 → c4 conjure up the same issues, but restriking the pitches renders the recursive interpretation slightly less disturbing than in c1 → c2. Just as the K-nets in Example 18 imply a very specific voice leading, the dual transpositions imply a very specific K-net interpretation. Note—by tracing the dashed arrows—that the recursive voices maintain the same partition, [04][05], throughout the passage, while the adjacency voices repartition the sets in each progression. I refer to these mid-progression isomeric reinterpretations of sets as K-net *double-emploi*. Example 19 shows a graphic version of the adjacency voices. In the example horizontal K-nets are transformations and vertical K-nets are reinterpretations or isomers. Motion into and out of each chord involves a structural reconfiguration.

In this musical passage, like many others, I find that both voice leading interpretations describe significant musical processes despite their contradictory characteristics. The recursive K-nets elegantly capture the middleground relation between the structure of the chords and the structure of the progression, but imply extremely abstract foreground voices. On the other hand, the dual transpositions capture the simple common-tone voices that underlie my foreground experience of the passage, but imply a rather complex series of graphic transformations and reinterpretations.

As I find value in both the concrete mappings of surface-reinforced voices and the often abstract mappings of network recursions, I frequently incorporate both in my analyses in the non-hierarchical levels labeled "Adjacencies" and "Recursive Structure" shown in Example 18. I call these levels non-hierarchical because they model different aspects of the music, and one interpretation does not subsume the next as in a Schenkerian graph. They do, however, still suggest varying degrees of distance from the musical surface. As a musician, I want to retain the perceptible musical surface in addition to modeling middleground relations. Merging graphic and dual approaches to transformational voice leading allows for this possibility.

*Example 18. Arnold Schoenberg,*
Three Piano Pieces, *Op. 11, No. 2, mm. 9–10.*



*Example 19. K-net* double emploi.