Ramsey Theory, Unary Transformations, and Webern's Op. 5, No. 4*

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This paper explores a collection of pitch-class sets that, although strikingly different by most measures, share a graph-theoretical property defined below. A refinement on this property for sets with seven to twelve elements isolates five set classes, two of cardinality 7 and three of cardinality 8, four of which are notable for their musical importance. These include the usual diatonic (7–35, in Forte's taxonomy of set classes), the octatonic set class (8–28), the diatonic-plus-a-fifth (8–23), and Messiaen's mode four (8–9), a set that plays a role in the movement by Webern discussed herein. The maverick in this group is the heptachord 7–29: {0124679}. Yet another refinement on the property uniquely isolates the usual diatonic and the octatonic, perhaps the most productive larger subsets of the aggregate.

This exercise forms part of an ongoing investigation into the abstract features of certain collections in the pitch domain that support more or less extensive musical repertoires. The diatonic collection is a celebrated case in point, and in the last few decades it has been studied from a variety of perspectives. Similarly, the highly symmetrical pitch-class sets, including Messiaen's modes of limited transposition, have received much attention, particularly for their importance in twentieth-century music. This paper studies a property that captures a number of musically significant sets and asks what else they have in common and how that might account for their musical potential.

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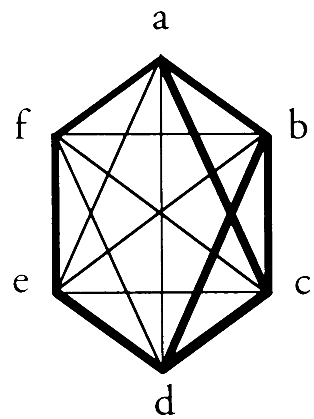
Ramsey Theory and its Application to Pitch-Class Sets

The property and its refinements arise in the simplest version of Ramsey theory. The eponymous Frank Ramsey was a mathematician who made remarkable contributions in a number of fields before his untimely death at the age of twenty-six in 1930. Let us begin with a simple puzzle arising from Ramsey's Theorem, a general and powerful result in mathematical set theory.[1] The puzzle asks, "What is the smallest party one can have, such that necessarily three of those present are either mutual acquaintances, or complete strangers?" Presumably, the existence of such triples is a good thing for parties. As it turns out, the smallest party where such a situation is necessarily the case has six in attendance.

This fact may be reformulated in terms of what graph theorists call chromatic complete graphs. A complete graph is a set of points, or *vertices*, arranged, let us say, in a circle, and all the line segments, or *edges*, that can be drawn between pairs of these points. A chromatic graph is one in which the edges are classified in some way, traditionally by assigning them various colors. Given a complete graph on six points, if the edges are colored in two ways, say, red and blue, somewhere in the graph there is a monochromatic triangle, with all sides either red or blue. This models the solution to the party puzzle, if we take blue to represent the relation of being acquaintances and red to represent being strangers, for instance. Actually, a stronger result can be proved: such a two-colored complete graph on six points has *two* forced monochromatic triangles. If there are only two monochromatic triangles, we will say the graph is *minimal*. Figure 1 presents a minimal configuration, where instead of coloring the edges red and blue they are marked by thick and thin line segments. It is possible to have a minimal configuration with two all-thick triangles, two all-thin triangles, or one of each kind.

[1]Frank P. Ramsey, "On a Problem of Formal Logic," *Proceedings of the London Mathematical Society* 2 (1930): 264-86.

Figure 1. Two-colored complete graph on six points, with a minimal solution to the party puzzle



monochromatic triangles:
(a, b, c) and (b, c, d)

Goodman solves the general problem of how many monochromatic triangles are forced in any two-coloring of the complete graph on n points. The minimum number depends on the value of n modulo 4.[2] Goodman's formulas: for n even, that is, n=2k, the number of forced monochromatic triangles is $\frac{1}{3}k(k-1)(k-2)$; if n=4k+1, the number is $\frac{2}{3}k(k-1)(4k+1)$; and if n=4k+3, the number is $\frac{2}{3}k(k+1)(4k-1)$. Thus, for n=7, there are a minimum of 4 monochromatic triangles, for n=8, there are 8, and for n=9, 10, 11 and 12 there are 12, 20, 28, and 40, respectively. Note that these equations correctly yield 0 when n is less than 6, and 2 when n=6.

Moreover, it is possible to have a minimal configuration where all the forced monochromatic triangles are of the same color for odd n *only* when n=7. I will call a configuration that has monochromatic triangles of only one color *pure*, here either pure

[2]A. W. Goodman, "On Sets of Acquaintances and Strangers at Any Party," *The American Mathematical Monthly* 69 (1959): 778-83.

all-thick or pure all-thin.[3] Thus, pure minimal configurations exist for odd n only for n=7. All even cardinalities permit pure minimal configurations.

Chromatic graphs in general and two-colored graphs in particular can model interpretations of pitch-class sets in various ways. The interval vector of a pitch-class set, for example, can be viewed as a tally taken on the complete graph of the set, with the edges sorted according to interval class (a six-colored chromatic graph). I will call an interval class *small* if it is 1, 2, or 3; *large* if it is 4, 5, or 6. Of course, small and large have no real meaning with respect to interval classes, but the distinction reflects the relative lengths of the line segments connecting points of a Krenek diagram, i.e., the relative sizes of the smallest representatives of the interval classes. If we assign blue (thick) to the small interval classes, and red (thin) to the large interval classes, we can apply the results for two-colored graphs. The monochromatic trichords divide into the *all-small* trichords (with a bow to Dr. Seuss) and the *all-large* trichord. The all-small trichords are the chromatic cluster type 3–1, prime form {012}, and the minor third type 3–2, {013}; the all-large trichord is the augmented triad 3–12, {048}. It is easy to see that the two all-small trichords are indeed the only ones with interval vectors containing entries only in the first three places, while the augmented triad is the only trichord-type with no entries in the first three places of its interval vector. With this interpretation, there are two set classes of cardinality 5 which yield critical graphs, that is, for which there are no all-small or all-large trichords: the usual pentatonic (5–35), and the other scalar arrangement of three major seconds and two minor thirds (5–34). (It is clear that we can associate an essentially unique graph to each set class, and that the classification does not depend on any particular representative of the set class:

[3]The mathematical literature uses terminology based on color, as I have indicated, and the term "blue-empty" is often encountered for graphs that have no all-blue triangles, but because I will be interested in interpretations of the colorings (as well as for reasons of practicality) I have altered the terminology to suit my purposes. A complete graph on 5 points that contains no monochromatic triangles is called a *critical* configuration.

transpositions and inversions of pitch-class sets correspond to rotations and reflections of graphs, and the count of monochromatic trichords remains undisturbed under these operations.)

It is by no means obvious that this all-small/all-large interpretation will yield any minimal configurations, but for most cardinalities it does. For hexachords, in fact, minimal cases are abundant: nineteen set classes are minimal, of which thirteen are pure all-small and two are pure all-large, i.e., the two monochromatic triangles in their graphs are of the same color. The complete list appears in Table 1. Among the minimal hexachords are many interesting ones, including four all-combinatorial hexachords, one of each order: the whole-tone set (6–35), the Guidonian hexachord (6–32), the hexatonic set (6–20), and type D (6–7); also the so-called Pétrouchka hexachord (6–30); Scriabin's mystic hexachord (6–34); and the Schoenberg signature hexachord (6–Z44). The minimal property may be relevant to serial composition in the case of hexachords, and I will return to this point briefly in the context of other interpretations of the Ramsey theory construction. In what follows, though, I will focus upon the larger sets, that is, sets of cardinalities 7 through 12, because of the overabundance of minimal hexachords and because my interest lies more with unordered sets that can serve to support individual compositions or even entire repertoires.

Among the set classes of larger cardinality there are also, it happens, nineteen minimal cases (none of which are pure all-large). The list of minimal sets for the larger cardinalities appears in Table 2. There are five minimal heptachord classes: among them are the usual diatonic 7–35, and set class 7–22, which as a scale in one ordering is sometimes referred to as Hungarian or gypsy minor (harmonic minor with raised $\hat{4}$). Various modes of this type of scale are used in the music of several cultures, as well as in Western art music.[4] There are ten minimal octachord

[4]For further discussion of Hungarian minor and of the special formal properties of 7–22 and its cognates, see David Clampitt, "Pairwise Well-formed

Table 1. Minimal hexachords

	Set class	pure all-small	pure all-large	trichords of same set class
6–Z6	{012567}	√		√
6–7	{012678}	√		√
6–18	{012578}	√		
6–20	{014589}		√	√
6–Z26	{013578}	√		√
6–Z29	{013689}	√		√
6–30	{013679}	√		√
6–31	{013589}			
6–32	{024579}	√		√
6–33	{023579}	√		√
6–34	{013579}			
6–35	{02468t}		√	√
6–43	{012568}	√		
6–Z44	{012569}			
6–Z46	{012469}	√		
6–Z47	{012479}	√		
6–Z48	{012579}			
6–Z49	{013479}	√		√
6–Z50	{014679}	√		√

Scales: Structural and Transformational Properties" (Ph.D. diss, State University of New York at Buffalo, 1997), 101-13 and Chapter 1, passim.

Table 2. Larger minimal sets

	Set class	pure all-small	all-small trichords of same set class
7–20	{0124789}		
7–22	{0125689}		
7–29	{0124679}	√	
7–30	{0124689}		
7–35	{013568t}	√	√
8–9	{01236789}	√	
8–16	{01235789}		
8–19	{01245689}		
8–20	{01245789}		
8–23	{0123578t}	√	
8–24	{0124568t}		
8–25	{0124678t}		
8–26	{0124579t}		
8–27	{0124578t}		
8–28	{0134679t}	√	√
9–12	{01245689t}		
10–5	{01234578te}		
10–6	{012346789t}		
12–1	{0123456789te}		

classes. The unique minimal 9-note set class, 9–12, is Messiaen's mode three. This is the complement of the augmented triad, and is the maximally even set of cardinality 9, as defined by Clough and Douthett.[5] There are two minimal 10-pc set classes: the set generated by ic5, and Messiaen's mode seven, the maximally even set that complements the tritone. The 11-pc set class is not minimal, and the aggregate is minimal. Of the ten set classes that support transpositional invariance and are of cardinalities 6 through 12, all are minimal.

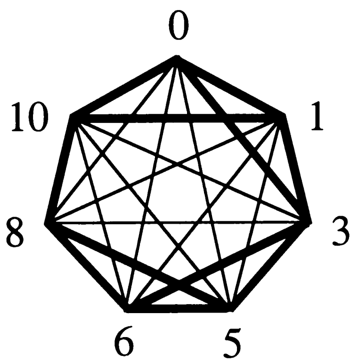
The first refinement on this property is to consider *pure* minimal sets. Only five set classes of cardinalities 7 through 12 satisfy this condition, shown in Figures 2 and 3. There are two pure minimal sets of cardinality 7: the usual diatonic set (generated by ic5 and maximally even) and 7–29, the only one among the pure minimal sets with distinct inversions. There are three pure minimal sets of cardinality 8: the octatonic (Messiaen's mode two and maximally even), 8–23 (generated by ic5), and 8–9, Messiaen's mode four. All of these are minimal pure all-small. Recall that 7 is the only odd cardinality that can support pure minimal sets.

The second refinement is to consider minimal pure all-small sets where the all-small trichords are of the same set class. The only larger sets that survive this filter are the diatonic and octatonic sets. In both cases, all of the all-small trichords are of the filled-in minor third type. In both the diatonic and octatonic, half of the all-small trichords are inverted forms (with respect to the other half).

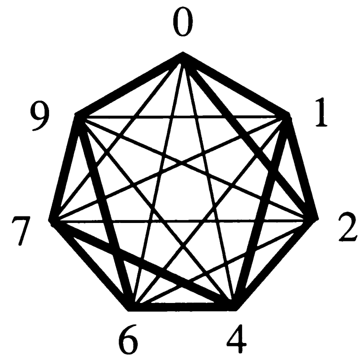
The suggestive lists of sets brought together on each of the three levels of this hierarchy are disparate, as might be expected, since memberships in the categories are determined solely according to properties of the trichordal subsets. What is being captured by this Ramsey theory construction? Another property, one more tightly circumscribed but also embracing both highly symmetrical sets and others with fewer degrees of symmetry,

[5]John Clough and Jack Douthett, "Maximally Even Sets," *Journal of Music Theory* 35/1-2 (1991): 93-173.

Figure 2. Minimal pure all-small 7-pc sets



7–35

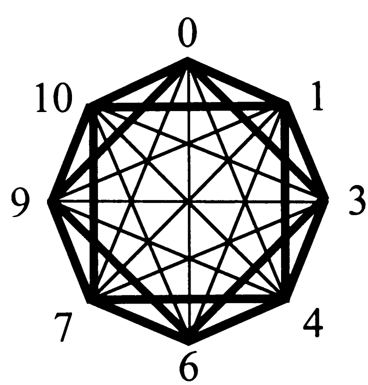


7–29

all-small trichords in 7–35:
(0, 1, 3) (3, 5, 6) (5, 6, 8) (t, 0, 1)

all-small trichords in 7–29:
(0, 1, 2) (1, 2, 4) (4, 6, 7) (6, 7, 9)

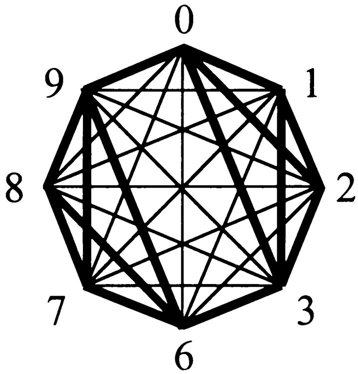
Figure 3. Minimal pure all-small 8-pc sets



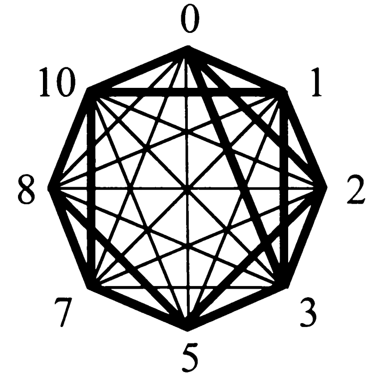
8–28

all-small trichords in 8–28:
(0, 1, 3) (1, 3, 4) (3, 4, 6) (4, 6, 7) (6, 7, 9) (7, 9, t) (9, t, 0) (t, 0, 1)

Figure 3. Minimal pure all-small 8-pc sets (cont.)



8–9



8–23

all-small trichords in 8–9:
(0, 1, 2) (1, 2, 3) (6, 7, 8) (7, 8, 9)
(0, 1, 3) (0, 2, 3) (6, 7, 9) (6, 8, 9)

all-small trichords in 8–23:
(0, 1, 2) (1, 2, 3) (0, 1, 3) (0, 2, 3)
(2, 3, 5) (5, 7, 8) (7, 8, t) (t, 0, 1)

that overlaps with the classes described here, is Clough and Douthett's maximal evenness. Note that each of the maximally even set classes from cardinalities 5 to 10 are minimal or critical (the pentatonic), and in particular that the diatonic and octatonic classes, both minimal pure all-small where the all-small trichords are of the same set class, are maximally even. The minimization of the {012} trichord has the effect of tilting towards evenness in larger minimal sets, but using Block and Douthett's measure of evenness some minimal sets are low on the evenness scale.[6] In general, the higher the Forte label, the greater the Block-Douthett evenness. Thus, 8–9 is a minimal pure all-small set class that scores rather low in evenness.

[6] Steven Block and Jack Douthett, "Vector Products and Intervallic Weighting," *Journal of Music Theory* 38/1 (1994): 21-41.

The sets generated by ic5 are also well represented among the minimal sets. As is the case with maximally even sets, there is exactly one of this type per cardinality. Furthermore, the classes coincide for cardinalities 5 and 7: maximally even sets that are also generated are among the well-formed scales defined by Carey and Clampitt.[7] The ic5-generated sets of cardinalities 5, 6, 7, 8, 10, and 12 are all minimal or critical, and those of cardinalities 6, 7, and 8 are minimal pure all-small. Since both maximally even and generated sets have at least one degree of symmetry, none of the minimal sets that fall into these two classes have distinct inversional forms.

This still does not go very far to answer the question of how (if at all) these formal properties in and of themselves signify an attractive musical potential. Without answering this question in any definitive way, I will suggest some approaches to the problem. One possibility is to consider the large vs. small partition for microtonal universes. This permits one to improve one's formal intuitions somewhat—for example, one finds that minimal configurations have a better chance of arising if the chromatic universe has cardinality congruent to 0 mod 4—but we do not have many musical intuitions to bring to bear on microtonal set classes. In the usual 12-note universe, one can consider other reasonable ways of dichotomizing the interval classes to determine two-colored graphs. Two other balanced ways of partitioning the interval classes are even vs. odd, and consonant vs. dissonant, that is, the partition into the sets {345} and {126}.

The even-odd partitioning may not seem very promising: it does not have the potential significance of consonance-dissonance, nor even that of smallness-largeness; moreover, possible minimal configurations would contain only all-even trichords. Since the difference of numbers of the same parity is even and the difference of numbers of opposite parity is odd, if all the members of a trichord are from the same whole-tone scale, the trichord is all-even, and if the members are not all from the

[7] Norman Carey and David Clampitt, "Aspects of Well-Formed Scales," *Music Theory Spectrum* 11/2 (1989): 187-206.

same whole-tone scale, then the trichord is not all-even, but has two odd entries and one even entry in its interval vector. Thus, all and only hexachords with three elements from the even whole-tone scale and three from the odd whole-tone scale have exactly two all-even trichords, i.e., are minimal pure all-even. There are twenty-two such hexachordal set classes. Here the relevance to serial construction is more apparent. For twelve-tone rows with hexachords from the twenty-two set classes, the aggregate is partitioned into hexachords in two ways (the usual partition of the row into hexachords according to order numbers, and into the even and odd whole-tone scales), with the elements of each partitioning distributed evenly over the hexachords of the other partitioning. Andrew Mead, in response to my presentation of these notions, described his application of the Ramsey idea to hexachords, using the even-odd distinction, an outgrowth of his mnemonic device for classifying pitch-class sets according to the whole-tone scale partition.[8] Given Mead's compositional/theoretical interests, the restriction to hexachords makes sense, and the even-odd partitioning is relevant to the pitch-class/order-number isomorphism, as described above. Moreover, one can introduce at least one refinement on the property, by considering hexachords where the two all-even trichords are of the same set class. I find, however, the small-large distinction more to my purposes, for one reason because for odd n no minimal configurations are possible under the even-odd interpretation (nor are there any critical configurations).

Consider, for example, a seven-note pitch-class set. The best we can hope for is a division into four even elements and three odd elements (or vice versa, of course). The number of all-even trichords in such a set is *five*, since the three odd elements form one all-even trichord, and there are four all-even trichords formed from the four even elements. Recalling that the number of monochromatic triangles in a minimal configuration for n=7 is four, we can see that it is impossible to find a minimal configuration under the odd-even interpretation for n=7. A

[8]Andrew Mead, "Pedagogically Speaking: A Practical Method for Dealing with Unordered Pitch-class Collections," *In Theory Only* 7/5-6 (1984): 54-66.

similar argument holds for any odd value.[9] On the other hand, for even n, it is trivial to find minimal cases: just as with hexachords, we simply look for sets where the number of even and odd elements is equal. Thus, if n=2m, there are m elements in each group. The number of trichordal subsets from each group of elements of like parity is the combinatorial number "m choose 3," or $m!/(m-3)!3! = m(m-1)(m-2)/(3)(2)(1)$. Since all of these trichords are all-even, and there are two groups, there are exactly 2 times m choose 3 or $\frac{1}{3}(m)(m-1)(m-2)$ all-even trichords. This is precisely the number of monochromatic triangles in a minimal configuration, as determined by Goodman's formulas cited above.

On the other hand, the consonant-dissonant interpretation has the consequence that minimal configurations minimize consonant triads: here the harmonic triad is the only trichord corresponding to a monochromatic triangle. Of course, this selects against just such set classes as the diatonic and octatonic. It suggests, however, another attitude to take towards minimal pure all-small sets. Rather than focus on what is being minimized, that is, trichords where the elements are maximally pushed together (3–1 and 3–2) or maximally pulled apart (3–12), perhaps we should take the point of view that in these cases the *mixed* trichords, among them the harmonic triads, are maximized.

The following analogy, based on a construction communicated informally by Norman Carey, is suggestive. In the usual diatonic set, the two specific varieties of every non-zero generic interval can be sorted according to multiplicity as being either *rare* or *common*. Thus, minor seconds are rare, major seconds are common; minor thirds are common, major thirds are rare; perfect fourths are common, augmented fourths are rare; and similarly for the complements of these intervals. One can rank the diatonic modes according to the proportion of common or rare diatonic intervals from each modal center. In D Dorian, all intervals from D are common, and the number of rare intervals

[9]The same argument also shows why there are no critical configurations under this interpretation. Any arrangement of five pitch classes would have at least three of the same parity, which would form an all-even trichord.

increases by one for each move along the circle of fifths outward from D. Thus, for Lydian and Locrian, half of the intervals are rare, while for Ionian and Phrygian, one-third of the intervals are rare, and for Mixolydian and Aeolian, only one-sixth of the intervals are rare. In Ionian and Phrygian, then, there is a balance of rare and common intervals from the modal center, and thus a balance of information and redundancy relative to the center. Naturally, the balance is tipped toward the common intervals. A mode with only "rare" intervals would necessarily be non-diatonic: D-E♭-F♯-G♯-A♭-B♭-C♯-(D). Carey refers to this mode as "The Gray Picture of Dorian."

Unary Transformations and Minimal Sets

Contextually defined transformations of pitch-class sets that globally leave all but one pitch class fixed (unary transformations) are significant in both tonal and post-tonal music.[10] One property common to the three 8-pc minimal pure all-small sets and several other minimal sets relates to unary transformations of some of their maximal proper subsets, particularly the Cohn functions described by Lewin.[11] In his 1994 paper, Cohn introduced the *P-relation*, which led to the *maximally smooth cycles* described by Cohn in a later article.[12] Two sets are P-related if there exists a transposition or inversion mapping one set to the other that leaves all but one pitch class of the sets invariant and moves the remaining pitch class by interval class 1. A P-cycle is a cycle of

[10]Forte introduced the term "unary transformation" in a paper entitled "New Modes of Linear Analysis," read at the Oxford University Conference on Music Analysis, Oxford, England, September 24, 1988.

[11]David Lewin, "Cohn Functions," *Journal of Music Theory* 40/2 (1996): 181-218.

[12]Richard Cohn, "Generalized Cycles of Fifths, Some Late-Nineteenth Century Applications, and Some Extensions to Microtonal and Beat-class Spaces." Keynote address for the Fifth Annual Meeting of Music Theory Midwest/Eighth Biennial Symposium of the Indiana University Graduate Theory Association, Bloomington, Indiana, May 14, 1994; Cohn, "Maximally Smooth Cycles, Hexatonic Systems, and the Analysis of Late-Romantic Triadic Progressions," *Music Analysis* 15/1 (1996): 9-40.

length greater than two where adjacent sets are P-related. The non-trivial P-cycles identified by Cohn in the 12-pc universe, shown in Figure 4, take place within the set classes of the harmonic triad, the usual pentatonic, the usual diatonic, and the complement of the harmonic triad, another interesting list.[13] A modified P-relation, which I will call a Q-relation, allows the single pc to move by any interval class, keeping the other conditions for a P-relation: preservation of set class, and maximal retention of common tones. Furthermore, for a proper Q-relation, I insist that the moving pc not "jump over" any of the stationary pcs. If there is a jump, I will refer to a Q*-relation. For example, in a set that will play a role in the Webern analysis, the 5–7 set class, {01267} and {01567} are Q-related, while {01267} and {0167e} are Q*-related.[14] In a Q-cycle adjacent sets are Q-related, with a special proviso for those trichords that may be described by the sequence of step intervals <abc>, with a, b, c distinct, (e.g., {013} trichords, described by the sequence of step intervals <129>). It is easy to see that such trichords support three possible Q-relations, but only two will be allowed to appear within a given Q-cycle. Thus, if the cycle proceeds by moving one pc to effect the exchange of intervals a and b, and then by moving another pc to effect the exchange of b and c, it may not also proceed by exchanging a and c.

The unary transformation that maps one pc-set in a set class to a Q-related member of the class is an instance of a *Cohn function*, a yet more general construction defined by Lewin. Q-cyclic set

[13]Enumerated in Cohn, "Maximally Smooth Cycles," 16. In Figure 4, the P-relations are conceptualized as contextually defined inversions, mapping adjacent sets to each other. The arrows here and in later figures are shown as unidirectional only to emphasize the notion of a cycle.

[14]The distinction between Q and Q* may be visualized in topological terms: a transformation between Q-related sets is smooth, involving stretching the set or sliding a note, whereas a transformation between Q*-related sets requires a tear, cutting a note from one place and pasting it elsewhere. The general distinction will be left here at the level of metaphor, but with the appropriate definitions in place, the distinction is a fundamental one in topology, which treats transformations of the former type, those that are smooth, i.e., "topological," as opposed to those of the latter type.

classes have a number of interesting concomitant properties, relating to organization in terms of generic and specific interval measure.[15] The Q-cycles themselves may be given multiple interpretations, depending on the analytical context.[16]

Figure 4. Cohn's non-trivial P-cycles

a. harmonic triad 3–11

$$\{037\} \xrightarrow{P} \{047\} \xrightarrow{P} \{e47\} \xrightarrow{P} \{e48\} \xrightarrow{P} \{e38\} \xrightarrow{P} \{038\} \xrightarrow{P} \{037\}$$

b. usual pentatonic 5–35

$$\{02479\} \xrightarrow{P} \{e2479\} \xrightarrow{P} \{e2469\} \ldots \xrightarrow{P} \ldots \{02579\} \xrightarrow{P} \{02479\}$$

c. usual diatonic 7–35

$$\{013568t\} \xrightarrow{P} \{013578t\} \ldots \xrightarrow{P} \ldots \{e13568t\} \xrightarrow{P} \{013568t\}$$

d. complement of harmonic triad 9–11

$$\{01235679t\} \xrightarrow{P} \{01235689t\} \ldots \xrightarrow{P} \ldots \{01245679t\} \xrightarrow{P} \{01235679t\}$$

[15]Described in David Clampitt, "Pairwise Well-formed Scales."

[16]For those familiar with the notion of a Generalized Interval System (GIS), defined in David Lewin, *Generalized Musical Intervals and Transformations*, (New Haven and London: Yale University Press, 1987), the pitch-class sets comprising a P- or Q-cycle can generally be construed as a GIS in three ways: as a commutative GIS with an associated cyclic group, and an interval function that measures intervallic distance in terms of distance along the P- or Q-cycle; and as a (generally) non-commutative GIS, with the associated group that acts on the sets in a simply transitive way construed either as a subgroup of the T_N/I_N group, or as a group of contextually defined operations. The triple description that these GIS structures afford underlies the transformational networks presented below in the Webern analysis. For more information and discussion, see Clampitt, "Pairwise Well-formed Scales," 24-36; idem, "Alternative Interpretations of Some Measures from *Parsifal*," *Journal of Music Theory* 42/2: 321-34; passim.

The three minimal pure all-small octachords each contain heptachords that support P- or Q-cycles. In the case of 8–23, since it is generated by the perfect fifth, two of its heptachordal subsets are diatonic sets, which participate in a P-cycle, as we have seen. The 7-note subsets of an octatonic set are all of the same set class, 7–31. As Figure 5 shows, these heptachordal subsets form a Q-cycle of length 8 (alternately P- and Q-related subsets: strictly speaking, the Q-relation includes the P-relation).[17]

Figure 5. The Q-cycle through the 7-pc subsets (of set class 7–31) of an octatonic set, given 8–28 represented by {0134679t}

$$\{0134679\} \xrightarrow{Q} \{t134679\} \xrightarrow{P} \{t034679\} \xrightarrow{Q} \{t014679\} \xrightarrow{P}$$

$$\{t013679\} \xrightarrow{Q} \{t013479\} \xrightarrow{P} \{t013469\} \xrightarrow{Q} \{t013467\} \xrightarrow{P}$$

$$\{0134679\}$$

The 7-note subsets of a representative of 8–9 fall into two classes of four: four each in 7–7 or 7–19. The four 7–7 sets participate in a Q-cycle as shown in Figure 6; the four 7–19 sets do not, but they pair up into two P-related couples, displayed in Figure 7. Finally, dropping the requirement that identity of set class be preserved but maintaining motion of a single pc by ic1 (I

[17]Again, these eight sets form the elements of a GIS in three ways. From one perspective that takes the existence of the Q-cycle as primary, the group acting on these sets is taken to be the cyclic group of order 8, and intervallic distance is measured in terms of the number of pitch classes moved or turned over as one proceeds through the cycle. From another perspective, the subgroup of order 8 of the full T_N/I_N group, consisting of elements T_0, T_3, T_6, T_9 and I_1, I_4, I_7, I_{10}, is construed as acting on the seven-note sets themselves, with the interval between sets related by I_1, for example, apprehended by the ear between all and only sets for which pcs 0 and 1 (or 6 and 7) exchange. (The same subgroup of order 8, but understood as acting on individual pitch-classes, also forms a GIS associated with the eight pitch classes of the octatonic pitch-class set {0134679t}.) The two contextually defined inversions also generate a group of order 8, acting on the heptachordal sets, where the interval is a measure of changes in configuration that take one heptachord to another.

will call this the R-relation), the eight 7-pc subsets of a representative of 8–9 participate in a cycle alternating P-, Q-, and R-relations, shown in Figure 8. All three relations are special cases of Forte's Rp.[18]

Figure 6. The Q-cycle through 7–7

$$\{0123678\} \xrightarrow{Q} \{0125678\} \xrightarrow{Q} \{012567e\} \xrightarrow{Q} \{014567e\} \ldots \xrightarrow{Q} \ldots$$
$$\{1236789\} \xrightarrow{Q} \{0123678\}$$

Figure 7. The two P-related pairs of set class 7–19 in 8–9, given 8–9 represented by {01236789}

$$\{0123679\} \xleftrightarrow{P} \{0123689\}$$
$$\{0136789\} \xleftrightarrow{P} \{0236789\}$$

Figure 8. The cycle of 7-pc subsets of 8–9: {01236789}

$$\{0136789\} \xrightarrow{P} \{0236789\} \xrightarrow{R} \{1236789\} \xrightarrow{Q} \{0123678\} \xrightarrow{R}$$
$$\{0123679\} \xrightarrow{P} \{0123689\} \xrightarrow{R} \{0123789\} \xrightarrow{Q} \{0126789\} \xrightarrow{R}$$
$$\{0136789\}$$

Table 3 summarizes the information on P- and Q-cyclic sets that are maximal proper subsets within minimal sets. Eleven of the nineteen larger minimal sets include subsets with cardinality one less that support cycles of unary transformations. Six of the eight non-trivial P- or Q-cyclic sets of cardinality 6 or greater are maximal proper subsets of one or more minimal sets.

[18] Allen Forte, *The Structure of Atonal Music* (New Haven and London: Yale University Press, 1973), 47.

Table 3. Immediate inclusion relations of P- and Q-cyclic sets within minimal and pure minimal sets of cardinalities 7 to 12

7–20	minimal	⊃ 6–Z44	Q-cyclic
7–22	minimal	⊃ 6–Z44	Q-cyclic
	(no other non-trivial Q-cyclic hexachords)		
8–9	minimal pure all-small	⊃ 7–7	Q-cyclic
8–16	minimal	⊃ 7–7	Q-cyclic
8–23	minimal pure all-small	⊃ 7–35	P-cyclic
8–26	minimal	⊃ 7–35	P-cyclic
8–27	minimal	⊃ 7–31	Q-cyclic
8–28	minimal pure all-small	⊃ 7–31	Q-cyclic
	(7–5 is the only other non-trivial Q-cyclic heptachord)		
10–5	minimal	⊃ 9–11	P-cyclic, ⊃ 9–5 Q-cyclic
10–6	minimal	⊃ 9–5	Q-cyclic
	(no other non-trivial Q-cyclic nonachords)		
12–1	(aggregate) minimal	⊃ 11–1	P-cyclic (trivially)

Transformations in Webern's Op. 5, No. 4

Webern's op. 5, no. 4 is an example of a work in which the set 8–9 functions as a background or source set, as confirmed in a number of analyses.[19] (This is, of course, one of the most often analyzed movements in the atonal literature.) We will see that networks and chains involving P- and Q-relations for subsets of 8–9 figure prominently. The unary transformations form the point of contact with the theoretical exposition of the Ramsey application, but otherwise the analysis is independent of that theoretical framework.

The heptachord 7–19 is the set class of the seven-note rising figure that Lewin calls *FLYAWAY*,[20] that articulates the ternary shape of the movement.[21] Figure 9 shows three Q-related 7–7 sets in op. 5, no. 4, exemplifying the "slow turnover of pitch material" remarked on by Beach.[22] The set classes 7–7 and 5–7 are the only non-trivial heptachord/pentachord complements that both support Q-cycles.[23] The Q- and Q*-related 5–7 sets in Figures 10 and 11 and displayed on the score in Example 1 show how this sonority and these common-tone relationships pervade

[19]Allen Forte, "A Theory of Set Complexes for Music," *Journal of Music Theory* 8/2 (1964): 136-83; David W. Beach, "Pitch Structures and the Analytical Process in Atonal Music: An Interpretation of the Theory of Sets,". *Music Theory Spectrum* 1 (1979): 7-22; Charles Burkhart, "The Symmetrical Source of Webern's Opus 5, no. 4," *Music Forum* 5 (1980): 317-34; Patricia Hall, "Letter to the Editor," *Music Theory Spectrum* 4 (1982): 163-67; Richard A. Kaplan, "Transpositionally Invariant Subsets: A New Set-Complex Relation," *Intégral* 4 (1990): 37-66; Richard S. Parks, "Pitch-class Set Genera: My Theory, Forte's Theory," *Music Analysis* 17/2 (1998): 206-26.

[20]Lewin, *Generalized Musical Intervals and Transformations*, 188-89.

[21]Most analysts have heard the form of the movement in terms of two (or three) introductory measures (characteristically ambiguous metrically and out-of-focus timbrally), followed by an A section. The seven-note rising figure in m. 6 is heard as either a concluding gesture or as a transitional figure leading into the contrasting B section of mm. 7-10, followed by the gesture again and a compressed and revoiced version of the canon from the A section, forming a varied repetition A', concluding again with the seven-note figure.

[22] Beach, "Pitch Structures," 19.

[23]The clusters 5–1 and 7–1 count as the trivial examples; the usual diatonic and pentatonic participate in P-cycles.

the outer sections of this movement. The Q-relation is the stronger relation, and is isolated in Figure 11. Each one of the last four sets in this chain is the T_1-transpose of the set two links prior to it, following the logical ordering of the sets (Figure 12). The Q-relation chain all but follows the chronological order of the piece (reversed only in the initial move; presently I will locate the structural downbeat of the movement at m. 4, point of departure for the chain); note also that the chain (exactly one quarter of a complete Q-cycle) ends when the turnover of pitch classes is complete, such that first and last sets are disjoint. The Q-chain thus functions in the much-commented-upon strategy of chromatic completion in the piece.

Figure 12 shows some of the intervallic relationships in the GIS of the standard T_N/I_N group, acting on the twenty-four sets of the 5–7 set class. In particular, the intervallic relationship (I_6) between the third and fourth sets in the chain is the same as that between the first and last sets in the chain. In both relations, the pitch classes 11 and 7 exchange: in the former, the sets differ only in these pitch classes, whereas in the latter, the sets differ completely in their pitch-class constituents. Note that the index of inversion advances by one for each move along the Q-cycle.[24]

[24]For students of Lewin's theory of Generalized Interval Systems, to pursue the multiple GIS descriptions discussed in notes 5 and 6 one might rewrite the transformations in Figure 12 in terms of the two distinct contextually defined transformations, Q_3 and Q_4, that participate in the chain of Q-related sets. Q_3 inverts a 5–7 set by preserving the (0167) tetrachord, sliding one note a minor third; Q_4 inverts a 5–7 set by preserving the (0156) tetrachord, sliding one note a major third. The two contextually defined inversions together generate a group that acts in a simply transitive way on the elements of the set class 5–7, just as the T_N/I_N group does. These groups are isomorphic, and exemplify aspects of Lewin's GIS theory for non-commutative GISs. Each group forms the group of interval-preserving operations for the GIS associated with the other. For example, observing the symmetrical role of the I_6 transformation in Figure 12, note that in the GIS associated with the T_N/I_N group, int({0156e},{01567}) = I_6 = int({045te},{12678}) = int(Q_4Q_3{0156e}, Q_4Q_3{01567}). (Left-functional orthography is the convention assumed here: apply Q_3 first.)

Example 1. Webern, op. 5, no. 4

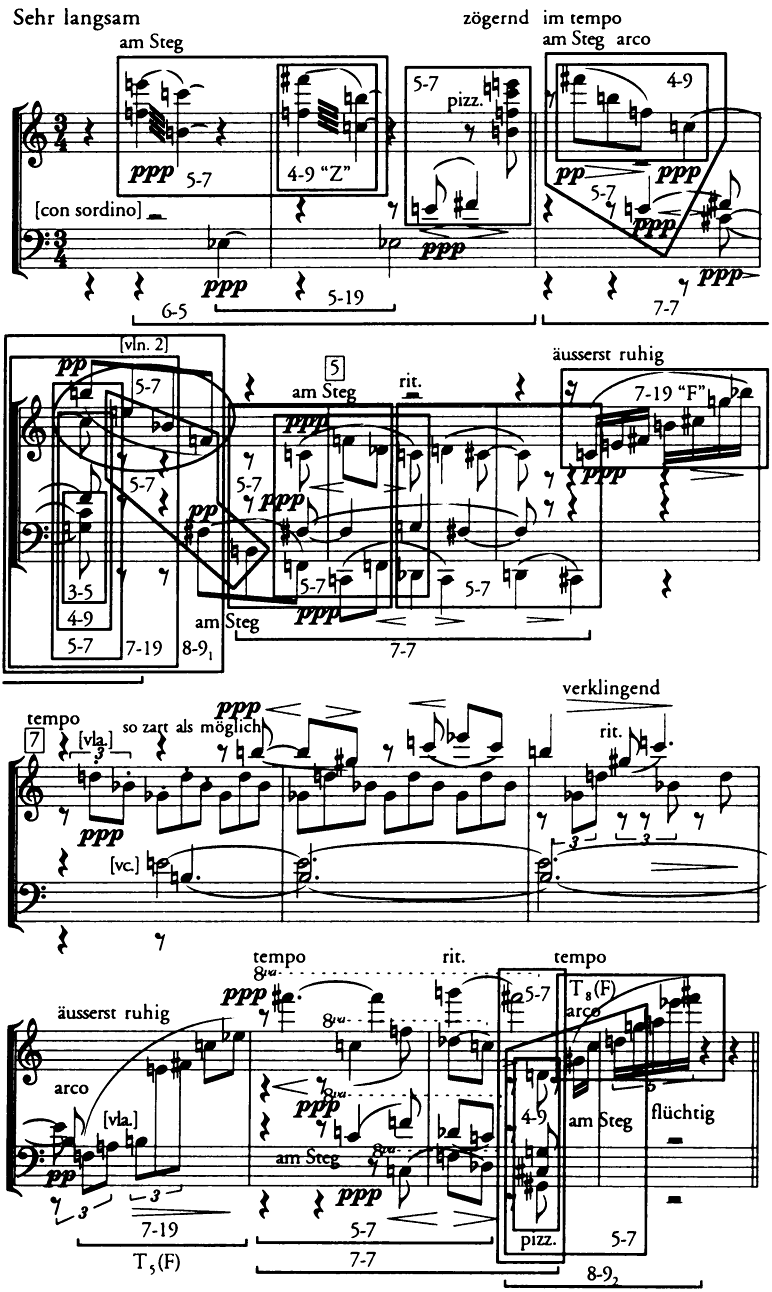


Figure 9. Three Q-related 7–7 sets in op. 5, no. 4

mm. 3-4/1 mm. 4/3-6/1 mm. 11-12/2

$$\{014567e\} \xrightarrow{Q} \{012567e\} \xrightarrow{Q} \{0125678\}$$

Figure 10. Q and Q relationships among ten 5–7 sets in op. 5, no. 4*

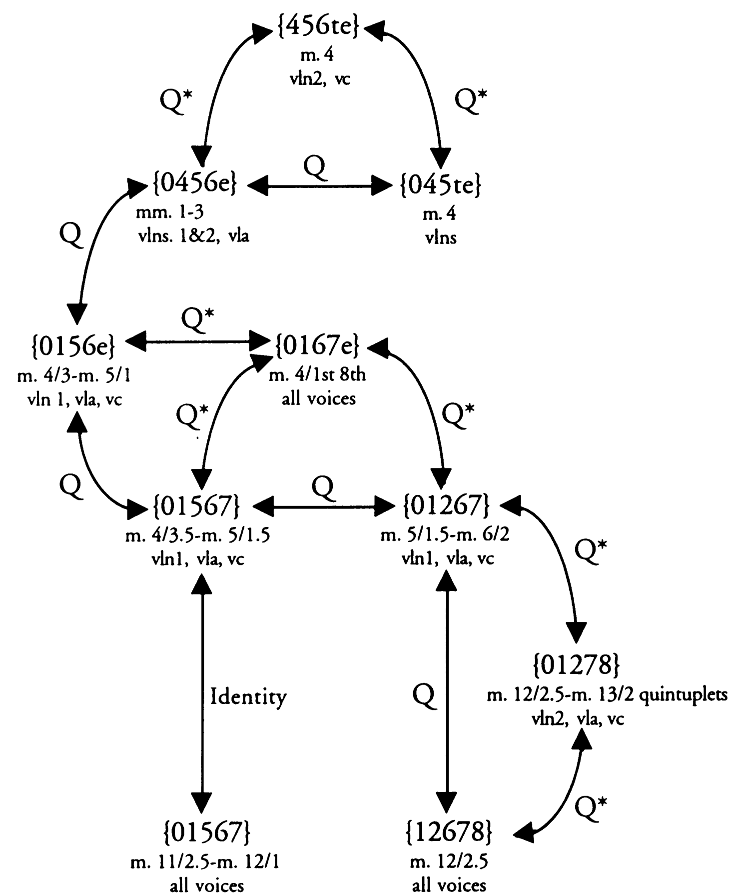


Figure 11. Chain of six Q-related 5–7 sets in op. 5, no. 4

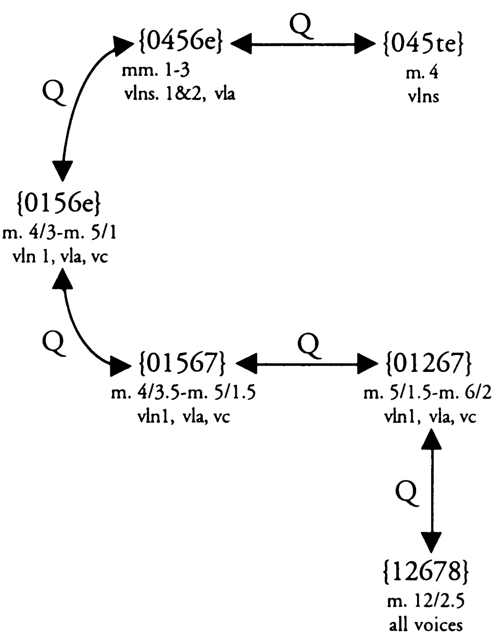
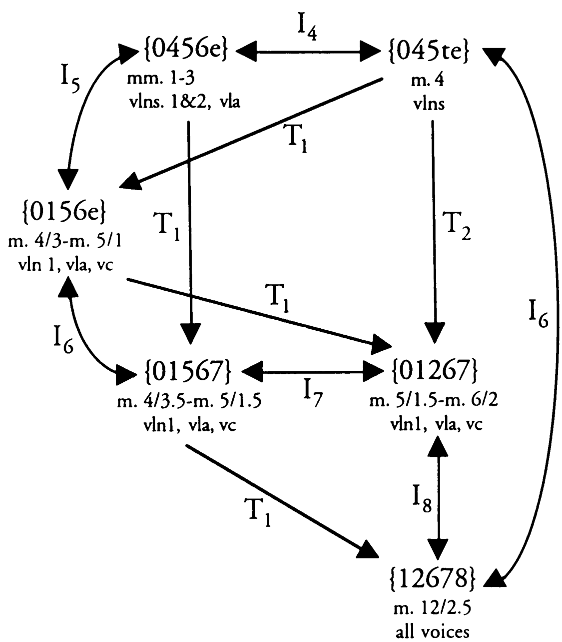


Figure 13 involves the relation of m. 4 to the subsequent appearances of the Flyaway gesture. I will give arguments from rhythm, pitch, and timbre for locating a structural downbeat in m. 4. The first observation concerns the metric ambiguity of the first three bars: there is no initial downbeat, nor is there a downbeat to m. 3; instead, a binary pulse is initially suggested by the alternation between the *tremolos* in the violins and the cello's E♭3. In m. 4, there are attacks on the downbeat in the outer voices. The soprano, here the second violin, is especially prominent because the texture clarifies, and because Webern marks it *äußerst zart*, differentiating it timbrally from the *am Steg* sound in the first three bars. Measures 4-5 then stabilize the triple meter, as Berry and Forte also observed.[25.]

[25] Wallace Berry, *Structural Functions in Music* (Englewood Cliffs, N. J.: Prentice-Hall, 1973; reprint, New York: Dover, 1980), 401; Allen Forte,

Figure 12. Network of six Q-related 5–7 sets in op. 5, no. 4, displaying selected relationships in the non-commutative GIS of T_N/I_N operators acting on the 5–7 set class



The argument from pitch concerns the articulation of various elements of the set complexes K, Kh, and Ki about 4–9/8–9.[26] (The roster of Kh members is 3–5, 5–7, 5–19, 6–5, 6–Z6/Z38, 6–7, 6–18, 6–30.) As the score shows, on the downbeat of m. 4, the lower voices play a form of set class 3–5, the sustaining voices play 4–9, and the whole quartet plays 5–7. The second violin plays a linear statement of 4–9, the T5 transform of the first

"Aspects of Rhythm in Webern's Atonal Music," *Music Theory Spectrum* 2:90-109 [100-105].

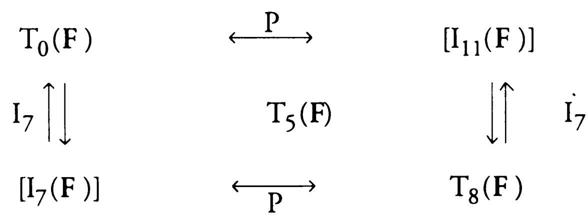
[26]The Ki set complex is defined in Kaplan, "Transpositionally Invariant Subsets."

violin's statement in m. 3, which itself linearizes the pitch content of the violins in m. 2 (labeled Z on the score).[27] The complete pitch content of m. 4 forms an 8–9 set, which may be seen as the union of two P-related forms of 7–19 (the set class of Flyaway). (See Figure 13.) The initial statement of Flyaway in m. 6 is a linearization of the content of the first three eighth-notes of m. 4. If one replaces the viola's F♯4 tied into the downbeat of m. 4 by the second violin's F♮4, the fourth eighth-note in the second violin's figure, that is equivalent to a shift from one form of 7–19 to another, its P-related "ghost," so to speak. If this segmentation seems capricious, note that F♯4 is the fixed element in a registral swap effected between the relevant pitches in m. 4 and m. 6: exclusive of the fixed F♯, the three highest pitches at the beginning of m. 4–C5, B5, and E5–are sent down one octave to form Flyaway in m. 6, while the three lower pitches–B♭4, C♯4, and G3–are sent up in register. The P-related forms of 7–19 map onto each other under I_{11}. The registral swap is *not* I_{11}, but F4/F♯4 is a semitone axis of inversion for a pitch-space inversion corresponding to I_{11}, and F4 and F♯4 are the notes that exchange in the P-relation. Note also that in the middle section, in mm. 7-9, the pitch set of the axial first violin melody maps onto itself under a pitch-space inversion corresponding to I_{11}. Finally, note that the form of Z (4–9) contained in the final statement of Flyaway, consisting of the notes G♯4, D5, A5, and E♭6, to be discussed below, again maps onto itself under a pitch-space inversion around an F5/F♯5 axis.[28]

[27]There is a tradition for calling the 4–9:{0167} tetrachord Z. When I refer to Z, I mean the pc-content of the violins in m. 2: {e056}. Other forms of Z will be labeled according to their transpositional/inversional levels.

[28]In a June 14, 1995 letter, David Lewin notes a registrally-ordered Flyaway form: D♭2 (cello m. 5), F2 (ibid.), G3 (viola m. 5), C4 (violin 1 m. 5), D4 (ibid.), and G♯5 and B5 (violin 1 mm. 7-8). (This is T_{11} of the m. 6 statement of Flyaway.) Lewin partially explains the detachment of the last two notes in this registral ordering by way of a "constructivist machine" discovered by Adam Krims, and refined by Lewin: One can generate the ordered Flyaway set by starting on what is to become its next-to-last note and cycling through the seven-note set produced by ascending 3, 2, 4, 2, 5, 2, 6 semitones, then rotating so that the first two notes become the last two notes.

Figure 13. Transformational network of forms of 7–19 in op. 5, no. 4



In the final two measures, two P-related forms of 7–19 are embedded in the I7- or T8-related form of 8–9 found there. In Figure 13, T0(F) is the initial form of 7–19 in m. 4 and is the content of Flyaway, and the P-related ghosts are shown in brackets. As the example shows, the relation between Flyaway, or F, and the ghost of T8(F) is the same as that between the ghost of F and T8(F); this relation is precisely I7.

The crucial instances of the gesture Flyaway have engaged the attention of several analysts, and in particular the question of why Webern chose the transpositional levels T5 and T8 arises. As Lewin remarks, a number of analysts have adduced "powerful set-theoretical rationales" for the viola's T5 transpositional level of Flyaway in m. 10.[29] For example, Perle has pointed out that the first four pitch classes in the first violin, C, E, F♯, B, appear as the first four notes of Flyaway.[30] If we call this unordered set Y, then notes 3 through 6 of Flyaway are T7(Y). Thus Y returns at its original transpositional level as notes 3 through 6 in T5(Flyaway). Lewin points out that not only are sets Y and Z themselves saturated by ic5, but also that ic5 spans between the sets Y and Z in three ways. I would add that the four lowest notes of the ostinato accompaniment in the middle section, B, E, B♭, G♭, form a Q*-related form of Y, and Z in m. 2 together with the E♭ in the

[29]David Lewin, "An Example of Serial Technique in Early Webern," *Theory and Practice* 7/1 (1982): 40-43 [41].

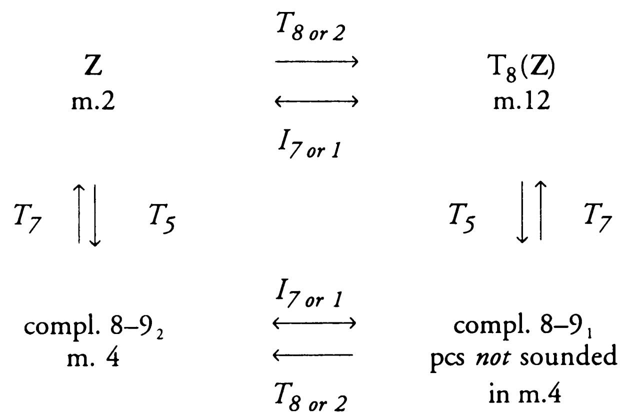
[30]George Perle, *Serial Composition and Atonality*, 2nd. ed. (Berkeley and Los Angeles: University of California Press, 1962), 16-19.

cello forms a 5–19 set, the included complement of T5(Flyaway).

Yet the final T8 level of Flyway has seemed both peculiarly right and yet difficult to account for. Comparing Figures 13 and 14, one can see that both I7 and T8 relations figure in these transformational networks. The T8 level of Flyaway that has seemed so problematic can be assimilated to the prominent I7 level, supplementing Lewin's rationale, which is based solely on the serial properties of Flyaway. For additional evidence of the prominence of the I7 level and its association with the T8 level, consider the pizzicato chord on the third eighth of m. 12. This chord is a form of Z, i.e., 4–9, and together with the content of m. 11 forms 7–7, in particular the I7 transform of the set encompassing the material in and around m. 5. The m. 12 chord itself is a form of Z from m. 2, but what form? I agree with Lewin's cogent rationale for considering it to be T8 of Z, in that it is a literal pitch transposition, and of course the T8(F) that follows immediately after it remains the primary question, but I think it is worth pointing out that this chord is also I7(Z), (as well as T2 and I1, because of Z's symmetries).

Figure 14 is a transformational network of forms of Z. The first statement of 8–9 in m. 4, labeled $8\text{–}9_1$, excludes pitch classes D, E♭, G♯, A. All of these pitch classes are included in the final statement of 8–9 (labeled $8\text{–}9_2$), and in fact are all included in T8(F), deployed registrally with inversional symmetry about the F5/F♯5 axis, as discussed above. This particular form of 4–9 never appears as a unit during the piece; instead, the first pair D-E♭ appears in the first half of the piece, E♭ as the second note and D as the next-to-last note in the first main section, and the chromatic is completed with the arrival of G♯ followed by A in the second half of the piece. The literal complement of $8\text{–}9_2$ does appear in the piece, however: the linear statement of T5(Z) in the second violin in m. 4.

Figure 14. Network of relations among forms of 4–9 in op. 5, no. 4



The tight coordination of set forms in this piece stems from the protean nature of the sets 4–9 and 8–9. The set {0,1,6,7} itself may be thought of as two semitone pairs separated by a tritone, or two tritones separated by a semitone, or two perfect fifths separated by a tritone. Thus, Boretz's analysis is organized about ordered pairs of perfect fifths separated by semitones, while Burkhart considers pairs of tritones separated by semitones, and Forte emphasizes the tritone saturation of the outer sections of the movement, by contrast with the ic4 emphasis of the middle section.[31] The complementary set has even greater versatility, in a sense that may best be seen by comparison with the octatonic set. The octatonic has a remarkable property, observed also by Cohn,[32] in that it may be partitioned into four disjoint pairs of any interval class. For example, the set {0134679t} contains four pairs spanning ic4: (0,4), (9,1), (3,7), (6,t). The set class 8–9 admits this property for all interval classes except ic3. In a representative of set class 7–19, then, three disjoint pairs may be formed for each interval class except ic3, and in each case one

[31]Benjamin Boretz, "Meta-variations, Part IV: Analytical Fallout (I)," *Perspectives of New Music* 11/1 (1972): 146-223 [217-23]; Burkhart, "The Symmetrical Source," 320-23; Forte, "A Theory of Set Complexes," 176.

[32]Richard Cohn, "Bartók's Octatonic Strategies," *Journal of the American Musicological Society* 44/2 (1991): 262-300 [271].

pitch class remains unpaired. The one note that satisfies these five unpaired pitch classes uniquely completes a set in set class 8–9.

Another way in which 8–9 forms a kind of one-off cousin to the octatonic 8–28 is the small size of its set complexes. Half of the eight members of its Kh complex figure in the analysis given above; only the octatonic has a smaller Kh complex. The small Kh complex size for these set classes is a reflection of how highly symmetrical they are. The pairing of these octachordal classes is noteworthy in light of the pervasive octatonicism that Forte discovers in Webern's atonal music.[33] It appears that Webern sometimes favored the octatonic's minimal pure all-small cousin. A movement where 8–28 and 8–9 confront each other and the analyst is the fourth of Webern's Bagatelles for String Quartet, op. 9. Forte makes the case for an octatonic reading of this movement, while Kabbash's segmentation suggests an orientation towards 8–9.[34]

The foregoing analysis has emphasized the importance of the highly symmetrical set class 8–9 in the fourth movement from Webern's op. 5. To return to the utility of minimal and pure minimal sets, the conditions imposed favor the selection of symmetrical sets. As we have seen, when the symmetry of each of the minimal pure all-small octachords and that of other minimal sets is disturbed by the removal of certain single notes, the resulting sets make available the sorts of strategies employed by Webern in this movement.

The picture is not a neat one, however, because not all of the minimal sets have Q-cyclic subsets, and not all of those that do are symmetrical. The Ramsey formulations generally provide only a measure of what kinds of configurations, and how many thereof, must occur within a sufficiently large universe of possibilities. The minimal criterion, although a global property of

[33]Allen Forte, *The Atonal Music of Anton Webern* (New Haven and London: Yale University Press, 1998).

[34]Forte, *Atonal Music of Webern*, 184-89; Paul Kabbash, "Aggregate-derived Symmetry in Webern's Early Works," *Journal of Music Theory* 28/2 (1984): 225-50 [226-32].

a pitch-class set, comes down to a count of closely packed and maximally spread-out trichords. One can infer a fair amount of pitch-class information on just the basis of the trichordal subsets, as the late Steven Gilbert's 1974 paper demonstrates,[35] and indeed the trichordal profile of a set, unlike the interval vector, is unique. Trichords form the basis for Forte's theory of pitch-class set genera.[36] In a formulation by Ayrey, cited by Forte in a response, "the trichords project their distinctive intervallic (and harmonic) properties through the universe of sets."[37] The Ramsey formulation projects a relative absence of 3–1 and 3–2 trichords, and of 3–12 trichords in combination with the all-small trichords. This leads to the complete exclusion of minimal sets from Forte's genus 5 (with trichordal progenitors 3–1 and 3–2), but surprisingly, perhaps, genus 4, derived from 3–12 alone, has on a proportional basis the largest constituency of minimal sets among the twelve genera. I omit further details on the distribution of minimal and/or pure minimal sets over the genera; its determination makes for an interesting exercise. The interaction between the various Ramsey applications and theories of pitch-class set genera is a possible avenue for futher investigation.

Whatever the interpretation chosen for the application of Ramsey theory, a somewhat impressionistic profile of the universe of pitch-class sets will necessarily be drawn. Many questions about the profile studied here remain, and "why" questions, such as, "Why are all of the larger transpositionally invariant sets minimal?" may not be susceptible to resolution. While it would not do to overestimate the application of this tool, Ramsey theory may be viewed as one more wedge with which to pry apart the complications of the world of tones.

[35]Steven Gilbert, "An Introduction to Trichordal Analysis," *Journal of Music Theory* 18/2 (1974): 338-62.

[36]Allen Forte, "Pitch-class Set Genera and the Origin of Modern Harmonic Species," *Journal of Music Theory* 32/2 (1988): 187-270.

[37]Craig Ayrey, "Berg's 'Warme die Lüfte' and Pc Set Genera: A Preliminary Reading," *Music Analysis* 17/2 (1998): 163-76 [167].