All Possible GZ-Related 4-Element Pairs of Sets, in All Possible Commutative Groups, Found and Categorized

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1. Preliminaries and definitions.

1.1 In an earlier work, I showed that 3-element "GISZ-related" sets cannot exist in a commutative GIS.[1] I call two sets "GISZ-related" if they are not GIS-transpositions or GIS-inversions, each of the other, and if they span the same assortment of GIS-intervals.

1.2 This paper finds and categorizes all possible conditions under which 4-element GISZ-related sets may exist in a commutative GIS. To simplify the discussion, we shall change the setting from a commutative GIS to a commutative group. We shall refer to "GZ-related" sets in the group, instead of "GISZ-related" sets in the GIS.[2]

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[1] David Lewin. 1997. "Conditions Under Which, in a Commutative GIS, Two 3-Element Sets Can Span the Same Assortment of GIS-Intervals; Notes on the Non-Commutative GIS in This Connection." *Intégral* 11: 37-66.

[2] In *Generalized Musical Intervals and Transformations* [henceforth *GMIT*] (New Haven: Yale University Press, 1987), I point out that the objects of any GIS can be labeled by the intervals of that System, in such a way that the interval from object s to object t is given by $\text{LABEL}(s)^{-1}\text{LABEL}(t)$ in the group of intervals (*GMIT*, 31). In a commutative GIS, using additive notation for the (commutative) group of intervals, the interval from s to t can then be given by LABEL(t) – LABEL(s). In Lewin 1997 (4.1.2 and fn. 16, p. 56), I point out that—conversely—any group G can be used as the family of formal objects for a GIS, if we take the group of formal GIS-intervals to be G itself, and define int(s,t) to be $s^{-1}t$. In a commutative GIS using additive notation for the (commutative) group, then, int(s,t) would be defined as t – s. The apparent leap in the main text here, from GIS theory to mathematical group theory, is thus not a weighty matter formally, despite the change in conceptual focus. Though motivated by music theory—specifically by a theoretical interest in GISZ-sets—the present paper can accordingly be read as a project in abstract mathematical group theory. So far as I know, the results reported are new music-theoretically as well as mathematically (I have consulted professional mathematicians in this regard). That is not surprising: mathematicians have not been much interested in musical "GISZ-sets," the corresponding mathematical "GZ-sets," or other kinds of "Z-sets" more generally.

1.3 **Definition:** Let G be a commutative group. Let S = {p, q, r, s} be a 4-element set of group elements. By the *n-transposition* of S we shall mean the set of elements {n + p, n + q, n + r, n + s}. By the *n-inversion* of S we shall mean the set of elements {n – p, n – q, n – r, n – s}.

1.4 **Definition:** Let G and S be as in (1.3). By the *interval-assortment* of S we shall mean the roster of the 12 non-zero "intervals" [q – p, p – q, r – p, p – r, r – q, q – r, s – p, p – s, s – q, q – s, s – r, r – s]. This is the roster of all non-zero group-differences that can be formed using the four members of S.

1.4.1 Some intervals may occur more than once on the roster of 1.4. For instance, in the additive group modulo 12, the set S = {0, 2, 5, 6} generates the roster [2, 10, 5, 7, 3, 9, 6, 6, 4, 8, 1, 11]. The interval 6 appears twice on this roster. In the same group, the set S = {0, 1, 2, 3} generates the roster [1, 11, 2, 10, 1, 11, 3, 9, 2, 10, 1, 11]. The intervals 1 and 11 each appear three times on this roster; the intervals 2 and 10 each appear twice.[3]

1.5 **Definition:** Let G and S = {p, q, r, s} be as in (1.3). Let V = {t, u, v, w} be a 4-element set of group elements from G. The sets S and V will here be called *GZ-related* when conditions 1.5.1 and 1.5.2 below both obtain:

1.5.1 V is neither an n-transposition nor an n-inversion of S (1.3), for any n in the group G.

1.5.2 V has the same interval-assortment (1.4), *en masse,* as does S.

1.5.3 **Example:** We saw in (1.4.1) that the set S = {0, 2, 5, 6} generates the roster [2, 10, 5, 7, 3, 9, 6, 6, 4, 8, 1, 11] in the additive group mod 12. Let V be the set {0, 5, 6, 8} within that group. We shall show that this S and this V are GZ-related. We compute the interval-assortment for V; it is [5, 7, 6, 6, 1, 11, 8, 4, 3, 9, 2, 10]. The two rosters match, *en masse*: each roster lists

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[3] Our "roster" idea is somewhat imprecise mathematically. For those readers who may be uncomfortable or curious, Lewin 1997 (1.2, p. 38) defines our "roster" more rigorously, as a certain mathematical function.

interval 1 once, interval 2 once, ..., interval 5 once, interval 6 twice, interval 7 once, ..., and interval 11 once. So (1.5.2) is satisfied. To see that (1.5.1) is also satisfied, we observe that S is "packed within a tritone"; hence any transposition or inversion of S can be "packed within a tritone." But V cannot be "packed within a tritone." So V cannot be any transposition or inversion of S.

1.5.4 For future reference, it will be useful to note some features that pertain to the particular mod-12 GZ-related sets S = {0, 2, 5, 6} and V = {0, 5, 6, 8} of 1.5.3 above.

1.5.4.1 S and V have 3 elements in common.

1.5.4.2 S can be expressed as {0, 6, 5 – 3, 5}, while V can be expressed as {0, 6, 5, 5 + 3}.

1.5.4.3 In this context, 2 = 5 – 3 while 8 = 5 + 3. We say that element 3 of the group is "of order 4": $4 \times 3 = 0$ mod 12, and $N \times 3 \neq 0$ mod 12 for any positive number $N < 4$.

1.5.4.4 The element 6 of the group is of order 2, and 6 is 2×3.

1.5.4.5 The element 5 of the group is not of order 2, and not a multiple of 3. Nor does $2 \times 5 = 2 \times 3$ mod 12.

2. Some further examples.

2.1 **Example:** Let G be the additive group mod 8. For musical application, we can use G to model the 8 "time-points" or "beat-classes" corresponding to the various eighth-notes of a 4/4 measure, labeling them as time-point 0 (at the opening bar line) through time-point 7 (just before the closing bar-line).

Let S be the set {0, 1, 3, 4}; let V be the set {0, 3, 4, 5}. We can think of S rhythmically, in our musical application, as "DEE duh - ta DAH - - -" (repeated indefinitely, measure after measure); we can think of V as "DEE - - ta DAH duh - -" (repeated indefinitely, measure after measure).[4]

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[4] It is interesting that the two figures are combined, using their three common time-points, in the figure that governs the first-movement Allegro of Chopin's Piano Sonata in B♭ minor: "DEE duh - ta DAH duh - -".

S can be packed within a span of 4 "eighth-notes"; V cannot be. Hence V cannot be any transposition or inversion of S. (1.5.1) is satisfied.

The interval-assortment of S is the roster [1, 7, 3, 5, 2, 6, 4, 4, 3, 5, 1, 7].[5] This roster lists twice the mod-8 intervals 1, 3, 4, 5, and 7; the roster lists 2 and 6 once each. The interval-assortment of V is the roster [3, 5, 4, 4, 1, 7, 5, 3, 2, 6, 1, 7]. This roster lists twice the mod-8 intervals 1, 3, 4, 5, and 7; the roster lists 2 and 6 once each. The two rosters are the same, *en masse*. So (1.5.2) is satisfied.

(1.5.1) and (1.5.2) both being satisfied, we conclude that *S and V here are GZ-related 4-element sets.*

2.1.1 For future reference, it will be useful to note some features that pertain to the particular mod-8 GZ-related sets S = {0, 1, 3, 4} and V = {0, 3, 4, 5} of 2.1 above.

2.1.1.1 S and V have 3 elements in common.

2.1.1.2 S can be expressed as {0, 4, 3 – 2, 3}, while V can be expressed as {0, 4, 3, 3 + 2}.[6]

2.1.1.3 In this context, 1 = 3 – 2 while 5 = 3 + 2. The element 2 of the group is "of order 4": $4 \times 2 = 0$ mod 8, and $N \times 2 \neq 0$ mod 8 for any positive number $N < 4$.

2.1.1.4 The element 4 of the group is of order 2, and 4 is 2×2.

2.1.1.5 The element 3 of the group is not of order 2, and not a multiple of 2. Nor does $2 \times 3 = 2 \times 2$ mod 8.

2.2 **Example:** Let G be the additive group mod 16. For musical application, we can use G to model the 16 "time-points" or

[5] We can "hear" *all* these intervals, in our rhythmic application, by allowing the figure to repeat indefinitely, measure after measure. In the ostinato, we will then hear the durations greater than 4, from one appearance of the figure, to its next appearance a measure later.

[6] The rhythmic figure from the Chopin sonata, "DEE duh - ta DAH duh - -", pointed out in fn. 4, supports this articulation of the overlapping time-point sets S and V in the ambient pitch structure. Time-points 0 and 4 of the common set {0,4} ("DEE" and "DAH") project the same pitch, D♭. Time-point 3—the "ta" common to both S and V—projects its own pitch, C. Time-point 1 = 3 – 2, the "duh" of set S, projects the pitch B♭; so does time-point 5 = 3 + 2, the "duh" of V.

"beat-classes" corresponding to the various sixteenth-notes of a 4/4 measure, labeling them as time-point 0 (at the opening bar line) through time-point 15 (just before the closing bar-line).

Let S be the set {0, 1, 5, 8}; let V be the set {0, 5, 8, 9}. We can think of S rhythmically, in our musical application, as "TEE tuh - - | - ta - - | TAH - - - | - - - -" (repeated indefinitely, measure after measure); we can think of V as "TEE - - - | - ta - - | TAH tuh - - | - - - - " (repeated indefinitely, measure after measure).

S can be packed within a span of 8 "sixteenth-notes"; V cannot be. Hence V cannot be any transposition or inversion of S. (1.5.1) is satisfied.

The interval-assortment of S is the roster [1, 15, 5, 11, 4, 12, 8, 8, 7, 9, 3, 13]. This roster lists twice the mod-16 interval 8; the roster lists 1, 3, 4, 5, 7, 9, 11, 13, and 15 once each; the roster does not list 2, 6, 10, or 14. The interval-assortment of V is the roster [5, 11, 8, 8, 3, 13, 9, 7, 4, 12, 1, 15]. This roster lists twice the mod-16 interval 8; the roster lists 1, 3, 4, 5, 7, 9, 11, 13, and 15 once each; the roster does not list 2, 6, 10, or 14. The two rosters are the same, *en masse.* So (1.5.2) is satisfied.

(1.5.1) and (1.5.2) both being satisfied, we conclude that *S and V here are GZ-related 4-element sets.*

2.2.1 For future reference, it will be useful to note some features that pertain to the particular mod-16 GZ-related sets S = {0, 1, 5, 8} and V = {0, 5, 8, 9} of 2.2 above.

2.2.1.1 S and V have 3 elements in common.

2.2.1.2 S can be expressed as {0, 8, 5 – 4, 5}, while V can be expressed as {0, 8, 5, 5 + 4}.

2.2.1.3 In this context, 1 = 5 – 4 while 9 = 5 + 4. We see that element 4 is "of order 4": $4 \times 4 = 0$ mod 16, and $N \times 4 \neq 0$ mod 16 for any positive number N smaller than 4.

2.2.1.4 The element 8 of the group is of order 2, and 8 is 2×4.

2.2.1.5 The element 5 of the group is not of order 2, and not a multiple of 4. Nor does $2 \times 5 = 2 \times 4$ mod 16 in the group.

3. A method of generating 4-element GZ-sets with 3 common elements, that works within some commutative groups.

3.1 We have now studied three specific examples of GZ-related 4-element sets that have 3 group-elements in common. In the three examples we see a pattern emerging from (1.5.4), (2.1.1), and (2.2.1).

In each case, we have a commutative group G, and an element g of order 4 within that group. (In 1.5.4, g = 3 mod 12; in 2.1.1, g = 2 mod 8; in 2.2.1, g = 4 mod 16.)

In each case, there is an element k of the group which is not of order 2, not a multiple of g, and which does not satisfy 2k = 2g. (In 1.5.4, k = 5 mod 12, which is not of order 2, not a multiple of 3, and does not satisfy $2 \times 5 = 2 \times 3$ mod 12. In 2.1.1, k = 3 mod 8, which is not of order 2, not a multiple of 2, and does not satisfy $2 \times 3 = 2 \times 2$ mod 8. In 2.2.1, k = 5 mod 16, which is not of order 2, not a multiple of 4, and fails to satisfy $2 \times 5 = 2 \times 4$ mod 16.)

In each case, we have a 4-element set $S = \{0, 2g, k - g, k\}$, and a 4-element set $V = \{0, 2g, k, k + g\}$. In each case, the sets S and V, which share the 3 common elements 0, 2g, and k, are GZ-related.

In the present section of this paper, we shall show that the above method of constructing GZ-related 4-element sets S and V will work in a certain class of commutative groups. It will work, specifically, in any commutative group that contains some element g of order 4, and also some element k that is not a multiple of g, and not of order 2, and does not satisfy 2k = 2g.[7]

[7] The method interacts cogently with the results of Stephen Soderberg in "Z-related Sets as Dual Inversions," *Journal of Music Theory* 39 (1995): 77-100. The work of our section 3, and of section 4 following, is less general in some ways than the work of Soderberg, and more general in other ways. It is less general because it concerns GZ-related sets of cardinality 4 only, and among those, only sets that have T-or-I related 3-element subsets. (Other 4-element GZ-related sets will be taken up in section 5 and following.) Soderberg's results apply to sets of cardinalities other than 4; his methods can also be used to find not just pairs of GZ-related sets, but also triples of such sets, etc. The work of our sections 3 and 4 is more general than Soderberg's in these respects: we have (in Theorem 4.3) completely necessary and sufficient conditions for 4-element sets with T-or-I

3.2 **Theorem:** *Let G be a commutative group that (i) contains an element g of order 4, and (ii) also contains an element k, not of order 2, that is not a multiple of g, and does not satisfy 2k = 2g. Define S to be the set {0, 2g, k – g, k}. Define V to be the set {0, 2g, k, k + g}.*

Then S and V are GZ-related 4-element sets with 3 common elements.

PROOF: Set S = {0, 2g, k – g, k} does indeed have 4 distinct members, and is therefore a "4-element set." 0 and 2g are distinct, since g is of order 4, not of order 2. k – g and k are each distinct from both 0 and 2g, since k is not a multiple of g. k – g is distinct from k, since g is not 0.

Likewise, V has 4 distinct members. Evidently S and V share the 3 common elements 0, 2g, and k.

To show that S and V are GZ-related, we must show (i) and (ii) below:

(i) V and S have the same interval-assortment.

(ii) V is not a transposition or an inversion of S.

3.2.1 Proof of (i) above.

Some intervals of S are formed by differences among the common elements 0, 2g, and k. Those intervals will necessarily match, *en masse*, the intervals of V formed by differences among those same common elements. So it suffices to show that the intervals of S formed by differences involving k – g are the same, *en masse*, as the intervals of V formed by differences involving k + g. We must therefore show that the roster [k – g, g – k, (k – g) – 2g, 2g – (k – g), (k – g) – k, k – (k – g)] matches, *en masse*, the roster [k + g, –(k + g), (k + g) – 2g, 2g – (k + g), (k + g) – k, k – (k + g)]. Writing "MEM" for "matches, *en masse*", we will show that [k – g, g – k, (k – g) – 2g, 2g – (k – g), (k – g) – k, k – (k – g)] MEM [k + g, –(k + g), (k + g) – 2g, 2g – (k + g), (k + g) – k, k – (k + g)].

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related 3-element subsets to be GZ-related, and it does so within commutative groups in all generality. Soderberg's results are formulated only for finite cyclic groups (integers mod N). In section 5 and beyond we shall discover and categorize all possible GZ-tetrads in all possible commutative groups; this goes farther than the tetrads that arise from Soderberg's Dual Inversions in finite cyclic groups.

Simplifying the algebra on those rosters, we must show [k – g, g – k, k – 3g, 3g – k, –g, g] MEM [k + g, –(k + g), k – g, g – k, g, –g]. k – g and g – k appear on both rosters; so do g and –g. It suffices to show that [k – 3g, 3g – k] MEM [k + g, –(k + g)]. And this is true because 3g = –g, g being of order 4.

Thus S and V do indeed have the same interval-assortment. We have now to show that V is neither a transposition, nor an inversion of S.

3.2.2 Proof of (ii) above.

3.2.2.1 We show first that V cannot be a transposition of S. The n-transposition of set S (1.3) is the set {n, n + 2g, n + k – g, n + k}. Suppose this set were in fact V = {0, 2g, k, k + g}. Then n, which is a member of S-transposed, would have to be a member of V = {0, 2g, k, k + g}. We distinguish four possible cases. Case 1: n = 0. Case 2: n = 2g. Case 3: n = k. Case 4: n = k + g. We take up each of these cases in turn, and show that none can happen.

Case 1: n = 0. Then S-transposed (by 0) is simply S. But S cannot be V. k – g is a member of S, and k – g is distinct from each of the common elements 0, 2g, and k. If S were the same as V, then k – g (in S) would have to be the same as k + g (in V). But in that case, –g would be g, and g would be of order 2, not order 4. This is not so. So Case 1 cannot happen.

Case 2: n = 2g. Then S-transposed, which is {n, n + 2g, n + k – g, n + k}, would be {2g, 0, g + k, 2g + k}. This would have to be the same set as V = {0, 2g, k, k + g}. Then 2g + k (of S-transposed) would have to equal k (of V). Then 2g would have to be 0, and g would be of order 2, not order 4. So Case 2 cannot happen.

Case 3: n = k. Then S-transposed, or {n, n + 2g, n + k – g, n + k}, would be {k, k + 2g, 2k – g, 2k}. This would have to be the same set as V = {0, 2g, k, k + g}. Then the element k + 2g, a member of S-transposed, would have to be some member of V, either 0 or 2g or k or k + g. But none of these can happen. k + 2g cannot be 0 because k is not a multiple of g. (2g = –2g.) k + 2g cannot be 2g because k is not 0. k + 2g cannot be k because 2g is not 0. (g is of order 4, not order 2.) k + 2g cannot be k + g because g is not 0. So Case 3 cannot happen.

Case 4: n = k + g. Then S-transposed, which is {n, n + 2g, n + k – g, n + k}, would be {k + g, k + 3g, 2k, 2k + g}. This would have to be the same set as V = {0, 2g, k, k + g}. Then the element k + 3g, a member of S-transposed, would have to be some member of V, either 0 or 2g or k or k + g. But none of these can happen. k + 3g cannot be 0 or 2g because k is not a multiple of g. k + 3g cannot be k because 3g is not 0. (g is of order 4.) k + 3g cannot be k + g because 2g is not 0. (g is not of order 2.) So Case 4 cannot happen.

Since none of Cases 1 through 4 can happen, V is not a transposition of S.

3.2.2.2 Now we show that V cannot be an inversion of S. The n-inversion of set S (1.3) is the set {n, n – 2g, n – k + g, n – k}. Suppose this set were in fact V = {0, 2g, k, k + g}. Then n, which is a member of S-inverted, would have to be a member of V = {0, 2g, k, k + g}. We distinguish four possible cases. Case 1: n = 0. Case 2: n = 2g. Case 3: n = k. Case 4: n = k + g. We take up each of these cases in turn, and show that none can happen.

Case 1: n = 0. Then S-inverted, i.e., {n, n – 2g, n – k + g, n – k}, would be {0, –2g, g – k, –k} = {0, 2g, g – k, –k}. If this were the same set as V = {0, 2g, k, k + g}, then {g – k, –k} would be the same 2-element set as {k, k + g}. –k cannot match k, since k is not of order 2. So –k must match k + g, and then g – k must match k. That is, we would have both –k = k + g and g – k = k. Then g = –2k, and also g = 2k. Then –g = g, and g would be of order 2, not order 4—which was supposed not the case. So Case 1 cannot happen.

Case 2: n = 2g. S-inverted, which is {n, n – 2g, n – k + g, n – k}, would be {2g, 0, 3g – k, 2g – k}. If this were the same set as V = {0, 2g, k, k + g}, then {3g – k, 2g – k} would be the same 2-element set as {k, k + g}. 3g – k cannot match k + g, for then 2g = 2k, which was supposed not the case. So 3g – k must match k, and then 2g – k must match k + g. We would have both 3g – k = k and 2g – k = k + g. Then 3g = 2k and also g = 2k. But then 3g = g, and 2g = 0, and g is of order 2, not order 4. So Case 2 cannot happen.

Case 3: n = k. Then S-inverted, or {n, n – 2g, n – k + g, n – k}, would be {k, k – 2g, g, 0}. If this were the same set as V = {0, 2g, k, k + g}, then {k – 2g, g} would be the same 2-element set as {2g, k + g}. But k – 2g cannot be 2g, since then we would have k = 4g = 0. Nor can k – 2g be k + g, for then we would have 3g = 0, and g would not be of order 4. So Case 3 cannot happen.

Case 4: n = k + g. Then S-inverted, which is {n, n – 2g, n – k + g, n – k}, would be {k + g, k – g, 2g, g}. If this were the same set as V = {0, 2g, k, k + g}, then {k – g, g} would be the same 2-element set as {0, k}. But neither k – g nor g can be 0. So Case 4 cannot happen.

Since none of Cases 1 through 4 can happen, V is not an inversion of S.

Thus 3.2.2.2 is proved; 3.2.2 is thereby proved; and Theorem 3.2 itself is proved.

3.3 **Corollary:** *Let G be a commutative group that (i) contains an element g of order 4, and (ii) also contains an element k, not of order 2, that is not a multiple of g, and does not satisfy 2k = 2g. Let S be the set OP({0, 2g, k – g, k}), where OP is some T-or-I operation. Let V be the set OP′({0, 2g, k, k +g}), where OP′ is some T-or-I operation.*

Then S and V are GZ-related 4-element sets containing respective 3-element subsets that are T-or-I related.

4. The converse of Corollary 3.3.

4.1 Corollary 3.3 gives us a very general method, in certain commutative groups, of constructing 4-element GZ-related sets that have T-or-I related 3-element subsets. We shall now prove a remarkable theorem: the method of (3.3) is in fact *exhaustive.* That is, if S and V are GZ-related 4-element sets in any commutative group, and if S and V have T-or-I related 3-element subsets, then S and V *must* have the form described in Corollary 3.3, for suitable choices of g, k, OP, and OP′.

4.2 **Theorem:** *Let G be a commutative group. Suppose that S and V are 4-element sets in G that are GZ-related, and that have T-or-I related 3-element subsets. Then there exist elements g and k of G, and T-or-I operations OP and OP′, such that*

(i) g is of order 4, and

(ii) *k is not a multiple of g, nor is k of order 2, nor does 2k equal 2g, and*

(iii) *S = OP({0, 2g, k – g, k}), and also V = OP′ ({0, 2g, k, k + g}).*

PROOF: By assumption, S and/or V can be transposed or inverted so as to have three common members. Let us imagine S and V already so transformed. By a further transposition of each, we may suppose that the group element 0 is one of the three common members. Let us provisionally denote by a and b the non-zero common members of S and V (suitably transformed as above). Then, for some group-elements s and v, we may suppose S to be expressed as {0, a, b, s}, while we may suppose V to be expressed as {0, a, b, v}.

By assumption, S and V have the same interval-assortment. Since the 3-element set {0, a, b} contributes the same intervals to the S-roster, as it does to the V-roster, the intervals of S that involve s must match, *en masse*, the intervals of V that involve v. That is (writing "MEM" for "match, *en masse*"), we must have [s, –s, s – a, a – s, s – b, b – s] MEM [v, –v, v – a, a – v, v – b, b – v].

Since S and V are different sets, v does not match s on these rosters. We distinguish three possibilities. Possibility 1: v matches –s. Possibility 2: v matches a – s or b – s. Possibility 3: v matches s – a or s – b.

4.2.1 Possibility 1: v matches –s, on the rosters [s, –s, s – a, a – s, s – b, b – s] and [v, –v, v – a, a – v, v – b, b – v]. We show that if this is so, then Theorem 4.2 is true.

Supposing v = –s, then –v = s, and the 4-element rosters [s – a, a – s, s – b, b – s] and [v – a, a – v, v – b, b – v] must MEM. v – a of the V-roster cannot equal s – a of the S-roster, since v and s are distinct. So we may distinguish three cases. Case 1: v – a of the V-roster equals a – s of the S-roster. Case 2: v – a of the V-roster equals b – s of the S-roster. Case 3: v – a of the V-roster equals s – b of the S-roster. We shall show that neither Case 1 nor Case 2 can occur, and that if Case 3 occurs, the theorem is true.

Case 1: v – a = a – s. Then a – v = s – a, and we can prune the 4-element rosters to 2-element rosters. Thus: [s – b, b – s]

MEM [v – b, b – v]. v – b cannot match s – b, since v and s are distinct. So v – b must match b – s, and b – v must match s – b.

In sum, we have v + s = 2a (the assumption of Case 1), and v + s = 2b (since v – b = b – s). But also v + s = 0 (the assumption of 4.2.1). So we have 2a = 2b = v + s = 0. In particular, –a would be a, and –b would be b. But then the 0-inversion of S, that is {–0, –a, –b, –s}, would be the set {0, a, b, v}, which is in fact V. And that contradicts the supposition that S and V are GZ-related. Case 1 cannot happen.

Case 2: v – a = b – s. Then v + s = a + b. But (4.2.1) has supposed that v = –s. Hence 0 = v + s = a + b. So b = –a. Then the 0-inversion of set S is {–0, –a, –b, –s}, which is {0, b, a, v}, which is V. But then, S and V being inversions of each other, the sets are not GZ-related. Case 2 cannot happen.

Case 3: v – a = s – b. We show that in this case the theorem is true. Now (4.2.1) has supposed that v = –s, and therefore that the 4-element rosters [s – a, a – s, s – b, b – s] and [v – a, a – v, v – b, b – v] must MEM. Since Case 3 supposes that v – a = s – b, we have a – v = b – s, and the 2-element rosters [s – a, a – s] and [v – b, b – v] must match, *en masse*. We distinguish Subcase 3.1: s – a = b – v; and Subcase 3.2: s – a = v – b.

Subcase 3.1: s – a = b – v. Then s + v = a + b. And the supposition of (4.2.1) is that v = –s. So 0 = s + v = a + b; v = –s and b = –a. Then the 0-inversion of set S is {–0, –a, –b, –s}, which is {0, b, a, v}, which is V. But then, S and V being inverted forms of each other; the sets are not GZ-related. Subcase 3.1 cannot happen.

Subcase 3.2: s – a = v – b. So v – s = b – a. But also v – s = a – b (the supposition of Case 3). Then v – s is its own negative. Let us set v – s = x. Then

(i) x is its own negative, and

(ii) x = v – s = s – v = b – a = a – b. But s = –v (4.2.1). So

(iii) x = v – s = s – v = 2v = 2s = b – a = a – b.

Since 2s = x and 2x = 0 (via (i)) but x is not itself 0, it follows that 4s = 0 but 2s is not 0. Hence

(iv) s is of order 4.

So v = –s (4.2.1) = 3s (by (iv) above). Then a ≠ 0, nor does a = s (since a and s are distinct within 4-element set S), nor does a = 3s (since a and v are distinct within 4-element set V). Nor can a = 2s, for then a = x (by (iii) above), and so –b = 0 (via (ii) above), and so b = 0, and b is not distinct from 0 within 4-element set S. Thus

(v) a is not a multiple of s.

If 2a were 0, then, since b – a = a – b ((iii) above), 2b would also be 0. Then a would be –a and b would be –b. Then the 0-inversion of S, the set {–0, –a, –b, –s}, would be {0, a, b, v}, which is V. So S and V would be inversions of each other, not GZ-related. So

(vi) 2a is not 0.

If 2a were x, then x – a would be a, and then, since x = a – b ((iii) above), a would be –b. Then the 0-inversion of S, the set {–0, –a, –b, –s}, would be {0, b, a, v}, which is V. So S and V would be inversions of each other, not GZ-related. So

(vii) 2a is not x; 2a is not 2s (via (iii) above).

Let us define g as s; let us define k as a. Then g is of order 4 ((iv) above), k is not a multiple of g ((v) above), nor is k of order 2 ((vi) above), nor does 2k equal 2g ((vii) above.

Let us define OP and OP′ both to be a-inversion. Then OP({0, 2g, k – g, k}) is the a-inversion of {0, 2s, a – s, a}, which is {a – 0, a – 2s, a – (a – s), a – a}, which is {a, a – x, s, 0}, which (via (iii) above) is {a, b, s, 0}, which is S. And OP′({0, 2g, k, k + g}) is the a-inversion of {0, 2s, a, a + s}, calculated as {a – 0, a – 2s, a – a, a – (a + s)}, which is {a, a – x, 0, –s}, which (by (iii) above) is {a, b, 0, v}, which is V. So all three requirements of the theorem obtain in this case. We have shown that, in Case 3, Subcase 3.2, the theorem is true.

4.2.2 Possibility 2: v matches a – s or b – s, on the matching rosters [s, –s, s – a, a – s, s – b, b – s] and [v, –v, v – a, a – v, v – b, b – v]. We show that this cannot happen.

Since the situation is symmetrical in the symbols a and b, we may assume that

(i) v matches a – s.

Then –v = s – a, and we can prune the rosters: [s, –s, s – b, b – s] MEM [v – a, a – v, v – b, b – v]. Since v is assumed to match a – s, then s must match a – v. And –s must match v – a. So we can prune the rosters even more: [s – b, b – s] MEM [v – b, b – v]. It follows that s – b cannot match v-b, since s and v are distinct. Hence

(ii) s – b matches b – v.

(i) above tells us that s + v = a. (ii) above tells us that s + v = 2b. All told, then,

(iii) s + v = a = 2b.

The a-inversion of set S is then the set {a – 0, a – a, a – b, a – s}, which, via (iii) above, is {a, 0, b, v}, which is set V. But then S and V are inversions of each other; they cannot be GZ-related. So Possibility 2 cannot happen.

4.2.3 Possibility 3: v matches s – a or s – b, on the matching rosters [s, –s, s – a, a – s, s – b, b – s] and [v, –v, v – a, a – v, v – b, b – v]. We show that if this is so, then Theorem 4.2 is true. Without loss of generality, we can suppose that

(i) v = s – a.

Then, pruning the matching rosters above accordingly, we see that [s, –s, s – b, b – s] MEM [v – a, a – v, v – b, b – v]. Matching these pruned rosters, we see that s must match one of v – a, a – v, v – b, or b – v. But s cannot match either a – v or b – v: this would entail the relationship "v = a – s or b – s," a relationship already ruled out as Possibility 2 above.

So s must match either v – a or v – b. We distinguish the two cases: Case 1: s = v – a; Case 2: s = v – b.

Case 1: s = v – a. In pruning the matching 4-element rosters [s, –s, s – b, b – s] and [v – a, a – v, v – b, b – v], we infer that [s – b, b – s] MEM [v – b, b – v]. However, s – b cannot equal v – b: s and v are distinct. Hence

(ii) s – b = b – v, and therefore v + s = 2b.

Then 2b = v + s (via (ii)) = v + (v + a) (via (i)) = 2v – a. So

(iii) a = 2(v – b).

Now Case 1 supposes that s = v – a. Then s = v – a = (s – a) – a (via (i) above), which equals s – 2a. Hence

(iv) 2a = 0.

Thus, by (iii) and (iv), 4(v – b) = 2a = 0. But 2(v – b), which equals a (iii), is not zero. Hence

(v) v – b is of order 4.

Set g = v – b, k = b. Then g is of order 4 ((v) above), v = b + g = k + g, and s = v – a (Case 1 assumption) = v - 2g (via (iii)), which is (k + g) – 2g, which is k – g. And a = 2g, via (iii).

Thus S = {0, a, b, s} = {0, 2g, k, k – g}, while V = {0, a, b, v} = {0, 2g, k, k + g}.

To show that Theorem 4.2 holds true in this event, then, we need only show that k is not a multiple of g, that k is not of order 2, and that 2k is not 2g.

We show that k is not a multiple of g. 0 and b are distinct members of S, so b is not 0. Thus k is distinct from 0 (which is 4g).

v and a are distinct members of V, so v must be distinct from 2(v – b) (via (iii) above), so 0 is distinct from v – 2b, so b is distinct from v – b; i.e., k is distinct from g.

b and a are distinct members of S, so b must be distinct from 2(v – b) (via (iii) above), so k is distinct from 2g.

v and 0 are distinct members of V, so 0 is not –v, so b is not b – v, so k is not –g. Since g is of order 4 (via (v) above), –g is 3g. So k is not 3g.

In sum, k is neither 0, nor g, nor 2g, nor 3g. k is, indeed, not a multiple of g.

We show that k = b is not of order 2. According to (ii) above, v + s = 2b. So if k = b were of order 2, we would have v + s = 0, and v would equal –s. 2b being 0, b would equal –b. Since a is of order 2 ((iv) above), a would be –a. Then the 0-inversion of S, which is the set {–0, –a, –b, –s}, would be {0, a, b, v}, which is V. S and V would be inversions of each other, and the sets would not be GZ-related. Hence k is not of order 2.

We show that 2k is not 2g. For if it were, then the 2g-inversion of set S would be {2g – 0, 2g – 2g, 2k – (k – g), 2k – k}, which is {2g, 0, k + g, k}, which is V. But then S and V would be inversions, each of the other, so S and V would not be GZ-related. Hence 2k is not 2g.

So Case 1 of Possibility 3 entails the truth of theorem 4.3.

Case 2: s = v – b. We shall show this cannot happen. Since Possibility 3 supposes that v = s – a, Case 2 implies that v – b = s = v + a, and b = –a, while v = s – a. We may then write S = {0, a, –a, s}, V = {0, a, –a, s – a}.

Suppose that S and V have the same interval-assortment. Then the intervals of S which involve s must MEM the intervals of V which involve s – a. So the roster [s, –s, s – a, a – s, s + a, –a – s] MEM the roster [s – a, a – s, s – 2a, 2a – s, s, –s]. Pruning the rosters, we see that [s + a, –a – s] matches, *en masse*, [s – 2a, 2a – s].

Thus either s + a = s – 2a (Subcase 1), or else s + a = 2a – s (Subcase 2).

In Subcase 1, where s + a = s – 2a, it follows that 3a = 0, and a is of order 3. Thus S = {0, a, 2a, s} while V = {0, a, 2a, s – a} = {0, a, 2a, s + 2a}. The 2a-transpose of set S is thus {2a + 0, 2a + a, 2a + 2a, 2a + s}, which is {2a, 0, a, s + 2a}, which is V. But then V, a transposition of S, cannot be GZ-related to S. So Subcase 1 cannot happen.

In Subcase 2, where s + a = 2a – s, it follows that s – a = –s. Then, while S = {0, a, –a, s}, V = {0, a, –a, s – a} = {0, a, –a, –s}. Hence V is the 0-inversion of S; the sets S and V are not GZ-related. So Subcase 2 cannot happen.

Then Case 2 cannot happen.

Theorem 4.2 is proved. We can sum up theorems 3.2 and 4.2 in one package.

4.3 **Theorem:** *Conditions (A) and (B) below are logically equivalent, for 4-element sets S and V in a commutative group:*

(A) S and V are GZ-related, and can be transposed or inverted so as to share a common 3-element subset.

(B) There exist elements g and k of the group, and transposition-or-inversion operations OP and OP′, such that

- *(i) g is of order 4.*
- *(ii) k is not a multiple of g, nor is k of order 2, nor does 2k = 2g.*
- *(iii) S = OP({0, 2g, k – g, k}), while V = OP′({0, 2g, k, k + g}).*

4.4 **Corollary:** Let G be a finite group in which 4-element sets S and V exist that are GZ-related and have T-or-I-related 3-element subsets. Then the cardinality of G must be divisible by 4.

PROOF: The "g" of (4.3) must be of order 4, and in any finite group (commutative or not), the order of any element will divide the cardinality of the group (this is a well-known result of group theory).

4.4.1 In addition to the finite groups of 4.4 above, there are also infinite groups that possess GZ-related 4-element sets of the type under discussion. The group of all non-zero complex numbers under multiplication is one such: the complex number i (the square root of −1) is of multiplicative order 4 in that group. The infinite group of complex numbers with absolute value 1, under multiplication, also contains i. So either of those infinite groups contains (infinitely many) 4-element GZ-related sets with T-or-I-related 3-element subsets.[8]

4.4.2 Even for the finite groups of 4.4 above, there will be many which are not cyclic (that is, are not of the form "integers modulo 4N" for some N). One that seems applicable to musical matters is the group G formed as the direct sum of groups G1 and G2, the latter groups each being a copy of integers-mod-4. G can be expressed mathematically as all ordered pairs <x,y>, x and y each being some integer mod 4. The group sum of <x,y> and <z,w> in G is the pair <x + z, y + w>.

In this group, the element g = <1,1> is of order 4, its multiples being <0,0>, <1,1>, <2,2>, and <3,3>. The element k = <1,2> is not a multiple of g, and 2k = <2,0> is not 0, nor is it 2g. So the sets S = {0, 2g, k − g, k} and V = {0, 2g, k, k + g} will be Z-related. These are the sets S = {<0,0>, <2,2>, <0,1>, <1,2>} and V = {<0,0>, <2,2>, <1,2>, <2,3>}. Numerous other such GZ-related 4-element sets may be formed, using other choices for g and k.

We observe that the group element k = <1,2>, in this example, is of order 4. It is of course not of order 2, nor is it a multiple <n,n> of g = <1,1>, nor is 2k = 2g (the former is <2,0>; the latter is <2,2>).

The group G of the present example may be applied musically to octatonic theory. The pairs of G can be used to label dyads of an octatonic scale for which one note belongs to one of the

[8] A typical GZ-related pair of such sets will have the form S = {x, −x, −ixk, xk} or S = {x, −x, ix/k, x/k}, V = {y, −y, yk, iyk} or V = {y, −y, y/k, −iy/k}, where k is some non-zero complex number (resp. of absolute value 1) other than 1, i, −1, or −i, and x and y are non-zero complex numbers (resp. of absolute value 1). Such sets are GZ-related in the pertinent *multiplicative* groups; that is, their assortments of *quotients* will MEM.

constituent diminished-seventh subsets, and the other note belongs to the other constituent diminished-seventh set. Thus in the octatonic set {C,C♯,E♭,E,F♯,G,A,B♭}, we may label the dyad {C,C♯} as <0,0>, the dyad {E♭,C♯} as <1,0>, the dyad {F♯,C♯} as <2,0>, …, the dyad {C,E} as <0,1>, the dyad {C,G} as <0,2>, …, the dyad {E♭,E} as <1,1>, the dyad {E♭,G} as <1,2>, …, the dyad {F♯,B♭} as <2,3>, …, and so forth.

In this representation, the elements of G can be used to label intervals between such note-pairs, subtracting the corresponding G-labels of the note-pairs. For instance, the interval from dyad {E♭,C♯} to dyad {F♯,B♭} is the interval from dyad <1,0> to dyad <2,3>, which is the difference (<2,3> – <1,0>) in group G, which is the element <1,3> of G.

The GZ-related numerical sets S and V above, S = {<0,0>, <2,2>, <0,1>, <1,2>} and V = {<0,0>, <2,2>, <1,2>, <2,3>}, label the octatonic dyad-sets S = {<C,C♯>, <F♯,G>, <C,E>, <E♭,G>} and V = {<C,C♯>, <F♯,G>, <E♭,G>, <F♯,B♭>}.

4.5 **Definition:** GZ-related 4-element sets in some commutative group that have the form of (4.3) above will be called "of form (4.3)."

5. GZ-related 4-element sets that are not of form (4.3).

5.1 Certain commutative groups may have GZ-related 4-element subsets that are not of the form described in (4.3) above.

5.1.1 For example, the 16-element group of (4.4.2) above possesses the 4-element sets S = {<0,0>, <1,0>, <0,1>, <3,3>} and V = {<0,0>, <1,0>, <0,3>, <3,1>}. We shall show that the sets are GZ-related, and that they are not of form (4.3).[9]

The (non-zero) interval-roster of S is [<1,0>, <3,0>, <0,1>, <0,3>, <3,1>, <1,3>, <3,3>, <1,1>, <2,3>, <2,1>, <3,2>, <1,2>].

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[9] Representing the group G as in (4.4.2), we can consider the sets S = {<0,0>, <1,0>, <0,1>, <3,3>} and V = {<0,0>, <1,0>, <0,3>, <3,1>} to be numerical labels for four-element sets of certain octatonic dyads. Under that identification, S = {<C,C♯>, <E♭,C♯>, <C,E>, <A,B♭>} and V = {<C,C♯>, <E♭,C♯>, <C,B♭>, <A,C♯>}.

The (non-zero) interval-roster of V is [<1,0>, <3,0>, <0,3>, <0,1>, <3,3>, <1,1>, <3,1>, <1,3>, <2,1>, <2,3>, <3,2>, <1,2>]. The sets S and V have the same interval-assortment.

V is not a transposition of S: a generic transposition of S, by the interval <m,n>, is S′ = {<m,n>, <m + 1, n>, <m, n + 1>, <m + 3, n + 3>}. Suppose S′ were the same set as V. Then one of the members of S′ would have to be <0,0>, a member of V. It is not possible for <m,n> to be that member of S′, since S is not the same set as V. It is not possible for <m + 1, n> to be the zero member of S′, since in that case <m,n> = <3,0>, which is a member of S′ but not of V. It is not possible for <m, n + 1> to be the zero member of S′, since in that case <m,n> = <0,3>, so that <m + 1, n> = <1,3>, being a member of S′, would have to be a member of V. But it is not. It is not possible for <m – 1, n – 1> to be the zero member of S′, since in that case <m,n> = <1,1> and <m,n>, while a member of S′, is not a member of V.

We inspect the generic <m,n>-inversion of S in the same manner. S′ = {<m,n>, <m – 1, n>, <m, n – 1>, <m + 1, n + 1>}, and we conclude that V is not any inverted form of S. So S and V above are not related by transposition or inversion. Having the same interval-assortment, they are then GZ-related.

But S and V here are not of form (4.3). A quick way to see this is to observe, by inspecting the interval-roster of S, that S spans no interval of order 2. But any GZ-related set of form (4.3) must span the interval "2g" of (4.3), an interval of order 2.

5.1.1.1 It is in fact a salient aspect of affairs here that S spans no interval of order 2. (Nor, of course, does V.) This feature will turn out to be typical of GZ-related tetrads that are not of form (4.3).

5.1.1.2 Even further: S does not span *any* non-zero interval in more than one way, as one may verify by inspecting its interval-roster. Again, this feature will turn out to be typical of GZ-related tetrads that are not of form (4.3).

5.1.1.3 Another salient aspect of affairs: there exist intervals a and b in the group G such that S can be expressed as the union of an a-dyad with a b-dyad, and V can also be expressed as the union of an a-dyad with a b-dyad. Here we take a = <1,0> and b = <3,2>. S is then the union of {<0,0>, <1,0>}, an a-dyad, with

{<0,1>, <3,3>}, a b-dyad, while V is the union of {<0,0>, <1,0>}, an a-dyad, with {<0,3>, <3,1>}, a b-dyad.

This feature is, however, *not* typical of (all) GZ-related tetrads that are not of form (4.3). The next example will illustrate the point.

5.1.2 Here is an example of GZ-related tetrads, not of form (4.3), that cannot be broken up into a-dyads and b-dyads in the manner of (5.1.1.3). In the additive group mod 13, the 4-element sets S = {0, 1, 4, 6} and V = {0, 2, 3, 7} are GZ-related. Their respective interval-assortments are [1, 12, 4, 9, 3, 10, 6, 7, 5, 8, 2, 11] and [2, 11, 3, 10, 1, 12, 7, 6, 5, 8, 4, 9]; these rosters each list every non-zero interval once, and so the rosters match, *en masse.* Additionally, the sets S and V above are not related by any T-or-I operation: S can be spanned within 6 consecutive numbers mod 13, while V, no matter how transposed or inverted, cannot be.

But S and V are not of form (4.3). To show this, we may invoke the group-theoretic theorem adduced in the proof of 4.4 above, to observe that the additive group mod 13 does not contain any element of order 4. Since the S and V mentioned just above are not of the form described in (4.3), they cannot have T-or-I-related 3-element subsets.

One verifies by inspection that S and V cannot both be expressed as an a-dyad and a b-dyad. For instance, S can be expressed as the 1-dyad {0,1}, union the 2-dyad {4,6}. But if we remove the 1-dyad of V, which is {2,3}, we are left not with a 2-dyad, but with the 6-dyad {0,7}. And so forth.

5.1.3 So there do exist GZ-related 4-element subsets of commutative groups, that do not have T-or-I-related 3-element subsets. What, in general, can we conclude about such pairs of 4-element sets? We begin such a study here. The first big push will take us eventually to Theorem 5.4.

5.2 **Theorem:** *Let S and V be GZ-related 4-element subsets of a commutative group that do not have T-or-I-related 3-element subsets [i.e., that are not of form (4.3)]. Then S cannot span any interval of order 2.*

PROOF: Suppose S did span an interval of order 2; suppose that interval is a. By assumption, the GZ-related V will also span interval a somehow. S and V may then be transposed into the forms S = {0, a, r, s}, V = {0, a, u, v}. After pruning the rosters slightly, we see that [r, –r, s, –s, r – a, a – r, s – a, a – s, s – r, r – s] must match, *en masse,* [u, –u, v, –v, u – a, a – u, v – a, a – v, v – u, u – v].

u cannot match r or s, since S and V are assumed to have no common 3-element subset.

u cannot match –r, since in that case the 0-inversion of S would be {–0, –a, –r, –s} = {0, a, u, –s} (–a being equal to a), and the 0-inversion of S would have 3 common elements with V, which is assumed by (5.2) not to be so. Symmetrically, u cannot match –s.

So, pruning the S-roster even more, we see that u must match something on the roster [r – a, a – r, s – a, a – s, s – r, r – s].

u cannot match a – r, since if it did, the a-inversion of S would be {a – 0, a – a, a – r, a – s} = {a, 0, u, a – s}, and that inversion of S would share 3 common elements with V, contrary to the assumption of (5.2). Symmetrically, u cannot match a – s.

So, pruning the S-roster even more, we see that u must match something on the roster [r – a, s – a, s – r, r – s].

u cannot match r – a, since if it did the a-transposition of S would be {a + 0, a + a, a + r, a + s}, which (since a = –a) is {a, 0, r – a, a + s}, which would be {0, a, u, a + s}, so that the a-transposition of S would have 3 common elements with V, contrary to supposition. Symmetrically, u cannot match s – a.

So, pruning the S-roster even more, we see that u must match something on the roster [s – r, r – s]. By the symmetry of the situation:

(i) We can suppose that u = s – r.

Going through the same procedure as above, trying to match v with something on the S-roster, we see that v must match something on the roster [s – r, r – s]; then, since u and v are distinct,

(ii) v must equal r – s, which is –u.

Going through the same procedure as above, trying to match r and s with some things on the V-roster, we see that r and s must match distinct things on the pruned roster [v – u, u – v]. We distinguish two cases. Case 1: r = v – u, s = –r; Case 2: r = u – v, s = –r. We shall show that neither of these two cases can occur, and that will prove Theorem 5.2.1.

Case 1: r = v – u. Via (i) and (ii) above, r = (r – s) – (s – r), which equals 2r – 2s. But if r = 2r – 2s, then r = 2s (via algebra). Since v = r – s ((ii) above), we would then have v = r – s = 2s – s = s. Since v = s, S and V would have 3 common elements, contrary to assumption. So Case 1 cannot happen.

Case 2: r = u – v. Invoking (i) and (ii) above, we can write r = u – v = (s – r) – (r – s) = 2(s – r) = 2u. Then u = s – r (via (i) above) = s – 2u, and 3u = s. Now r and s are negatives, each of the other (via (ii) applied to r and s), and we have just seen that r = 2u, while s = 3u. Hence

(iii) u is of order 5, and v (which is –u via (ii)) is therefore 4u.

Substituting 2u for r, 3u for s, and 4u for v, we can then write

(iv) S = {0, a, 2u, 3u}, V = {0, a, u, 4u}; u is of order 5.

The S-roster is then [a, a, 2u, –2u, 2u – a, a – 2u, 3u, –3u, 3u – a, a – 3u, u, –u]. (a is of order 2, hence equals –a.) And the V-roster is [a, a, u, –u, u – a, a – u, 4u, –4u, 4u – a, a – 4u, 3u, –3u]. Now since u is of order 5, 4u = –u and –4u = u. Accordingly, we may rewrite the V-roster above as [a, a, u, –u, u – a, a – u, –u, u, –u – a, a + u, 3u, –3u]. Pruning 2 a's, one u, and one –u off each roster, we see that the pruned S-roster [2u, –2u, 2u – a, a – 2u, 3u, –3u, 3u – a, a – 3u] must MEM the pruned V-roster [u – a, a – u, –u, u, –u – a, a + u, 3u, –3u].

Now the latest-pruned V-roster still lists one u, so u must match some member of the latest-pruned S-roster [2u, –2u, 2u – a, a – 2u, 3u, –3u, 3u – a, a – 3u].

Since u is of order 5, u cannot match 2u, nor –2u (which is 3u), nor 3u, nor –3u (which is 2u). So u must match something on the further-pruned roster [2u – a, a – 2u, 3u – a, a – 3u].

In any of those four events, we can show that a must equal some non-zero multiple of u. But a is of order 2, and every non-zero multiple of u is of order 5. So Case 2 cannot happen.

This proves Theorem 5.2.

5.3 **Lemma:** *In a commutative group, let S be a 4-element set spanning no interval of order 2. Suppose S spans some interval a in at* *least 2 different ways. Then S may be transposed or inverted so as to assume either Form (i) or Form (ii) below.*

Form (i): S = {0, a, 2a, x}, for some element x.

Form (ii): S = {0, a, y, y + a}, for some element y.

PROOF: Given the interval a as in the statement of the theorem, we can transpose S into the form {0, a, r, s}, for some r and s in the group. The interval-assortment of S is then [a, –a, r, –r, s, –s, r – a, a – r, s – a, a – s, s – r, r – s]. By supposition, a appears at least twice on this roster, and a does not appear as –a (since S spans no interval of order 2). Since S has 4 distinct elements, a does not appear on the roster as r, nor does it appear as s. a can not appear as a – r, since r is distinct from 0. Likewise, a cannot appear as a – s. So a matches something on the pruned roster [–r, –s, r – a, s – a, s – r, r – s]. We show that, no matter which of the four roster entries a matches, S is either already in one of the desired Forms of the Lemma, or can be transposed into one of those Forms.

If a = –r, then the a-transpose of S is {a + 0, a + a, a + r, a + s} = {a, 2a, 0, a + s}, and this transposed form of S is in Form (i) of the Lemma. Symmetrically, if a = –s, then the a-transpose of S is in Form (i) of the Lemma.

If a = r – a, then r = 2a and S = {0, a, r, s} = {0, a, 2a, s} is already in Form (i) of the Lemma. Symmetrically, if a = s – a, then S is already in Form (i) of the Lemma.

If a = s – r, then s = r + a and S = {0, a, r, s} = {0, a, r, r + a} is already in Form (ii) of the Lemma. Symmetrically, if a = r – s, then S is already in Form (ii) of the Lemma.

The Lemma is proved. We are ready now to prove a very general theorem.

5.4 **Theorem:** *In a commutative group, let S and V be GZ-related 4-element sets that have no T-or-I-related 3-element subsets [i.e., that are not of form (4.3)]. Then no non-zero interval appears more than once in the interval-assortment of S and V.*

PROOF: We know from (5.2) that no interval of order 2 appears among those of the interval-assortment.

Suppose now that some other interval a appears more than once in the interval-assortment. From (5.3), then, we infer that S (suitably transposed or inverted) must be of Form (i) or Form (ii) described therein, and V (suitably transposed or inverted) must also be of Form (i) or Form (ii).

It cannot be the case that both S and V (suitably transposed or inverted) are of Form (i). For if S = {0, a, 2a, x} and V = {0, a, 2a, y}, then S and V share a common 3-element subset, contrary to assumption. So either S or V must be of Form (ii). We shall suppose that V (at least) is of Form (ii), supposing that V = {0, a u, u + a}. We distinguish two cases. Case 1: S is of Form (i); Case 2: S is of Form (ii). We show that neither of these Cases can happen, and hence that there can be no such interval a.

Case 1: S (suitably transformed) is of Form (i). We have supposed that V = {0, a, u, u + a}; we shall suppose that S = {0, a, 2a, s}. The interval-assortment of S is then [a, –a, 2a, –2a, a, –a, s, –s, s – a, a – s, s – 2a, 2a – s]. And the matching-*en-masse* interval-assortment of V is [a, –a, u, –u, u – a, a – u, u + a, –a – u, u, –u, a, –a]. Pruning two a's and two –a's off each list, we see that [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s] MEM [2a, –2a, u, –u, u – a, a – u, u + a, –a – u, u, –u]. So u must match something on the list [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s]. We shall show this cannot be, so that Case 1 cannot happen.

u cannot match 2a, since S and V would then have in common the 3-element subset {0, a, 2a}.

u cannot match –2a, for then the a-transposition of V would be {a + 0, a + a, a + u, a + (u + a)} = {a, 2a, –a, 0}, and that would have 3 common elements with S (which is {0, a, 2a, s}).

u cannot match s, for then S = {0, a, 2a, s} and V = {0, a, u, u + a} = {0, a, s, s + a} would have 3 common elements.

We leave for a while the possibility that u might equal –s, and continue along [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s], from s – a on.

u cannot match s – a, for then the a-transposition of V would be {a + 0, a + a, a + u, 2a + u} = {a, 2a, s, s + a}, and that would have 3 common elements with S (which is {0, a, 2a, s}).

u cannot match a – s, for then the a-inversion of S would be {a – 0, a – a, a – 2a, a – s}, which is {a, 0, –a, u}, a set having 3 common elements with V (which is {0, a, u, u + a}).

We leave for the time being the possibility that u might equal s – 2a, and continue along [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s], from 2a – s on.

u cannot match 2a – s, for then the 2a-inversion of S would be {2a – 0, 2a – a, 2a – 2a, 2a – s} = {2a, a, 0, u}, which has 3 common elements with V = {0, a, u, u + a}.

Summing up, we see that there are only two entries on the roster [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s] which u could possibly match, namely u = –s and u = s – 2a. But that roster must not only list one u somewhere, it must in fact list u *twice*, since it must match, *en masse*, the roster [2a, –2a, u, –u, u – a, a – u, u + a, –a – u, u, –u], as described in the first paragraph of this discussion of Case 1.

Since the roster [2a, –2a, s, –s, s – a, a – s, s – 2a, 2a – s] must list u twice, and u cannot match anything on the roster except for (possibly) –s or s – 2a, we infer that u does equal –s, and *also* that u = s – 2a. But then –s = s – 2a, and 2(s – a) = 0. So s – a is of order 2. But s – a is an interval spanned by S, and (5.2) tells us that S can span no interval of order 2. Case 1 cannot happen.

Case 2: S (suitably transformed) is of Form (ii). We have supposed that V = {0, a, u, u + a}; we shall suppose that S = {0, a, r, r + a}. The interval-assortment of S is then [a, –a, r, –r, r – a, a – r, r + a, –a – r, r, –r, a, –a]. And the matching-*en-masse* interval-assortment of V is [a, –a, u, –u, u – a, a – u, u + a, –a – u, u, –u, a, –a].

Pruning the rosters, we see that [r, –r, r – a, a – r, r + a, –a – r, r, –r] must therefore match, *en masse*, [u, –u, u – a, a – u, u + a, –a – u, u, –u]. In particular, u must match something on the roster [r, –r, r – a, a – r, r + a, –a – r, r, –r].

u cannot equal r, for S and V have no common 3-element subset.

u cannot equal –r, for in that event we would have V = {0, a, u, u + a} = {0, a, –r, –r + a}, and the r-transpose of V would then be {r, r + a, 0, a}, which is S itself.

u cannot equal r – a, for in that event r = u + a, and the sets S and V would share the 3 elements 0, a, and r = u + a in common.

u cannot equal a – r, for in that event the a-inversion of S, which is {a – 0, a – a, a – r, a – (r + a)}, would be {a, 0, u, u + a}, which is V itself.

u cannot equal r + a, for in that event S and V would share the common 3-element subset {0, a, r + a} = {0, a, u}.

u cannot equal –a – r, for in that event the r-transpose of V would be {r + 0, r + a, r + u, r + u + a} = {r, r + a, r + u, 0}, which has 3 common elements with S = {0, a, r, r + a}.

But then, contrary to supposition, u does not match anything on the roster [r, –r, r – a, a – r, r + a, –a – r, r, –r]. So Case 2 cannot happen.

This establishes theorem (5.4).

5.4.1 **Corollary:** *In a commutative group, let S and V be GZ-related 4-element sets that have no T-or-I-related 3-element subsets [i.e., that are not of form (4.3)]. Then the group must be of cardinality 13 or greater (possibly infinite).*

PROOF: Via (5.4), the group must admit at least 12 distinct non-zero intervals, for 12 such will appear on the interval-assortment of S.

6. GZ-related tetrads, not of form (4.3), that can both be expressed as the union of an a-dyad and a b-dyad.

6.1 (5.1.1) above gave an example of a group G that contained 4-element GZ-related sets S and V, not of form (4.3), where it was the case that for suitable intervals a and b, both S and V could be expressed as the disjoint union of an a-dyad and a b-dyad. This was noted in 5.1.1.3.

(5.1.2) above gave an example of a group G that contained 4-element GZ-related sets S and V, not of form (4.3), where it was *not* the case that for suitable intervals a and b, both S and V could be expressed as the disjoint union of an a-dyad and a b-dyad. This was noted at the end of the section discussing that example.

In sum, of the GZ-tetrads-in-commutative-groups that are not of form (4.3), some have the property that, for suitable intervals a and b, both S and V can be expressed as the disjoint union of an a-dyad and a b-dyad. And some do not have that property. In this section we shall explore those that *do* have the indicated property.

We shall see that these tetrads all have the following structure:

(i) The group includes (an isomorphic copy of) the 16-element group of example (5.1.1) as a subgroup.

(ii) The GZ-tetrads S and V of the big group are transpositions-or-inversions (within the big group) of the GZ-tetrads S and V of (5.1.1), as those are expressed within the 16-element subgroup described in (i) above.

These informal observations are expressed more precisely in the theorem that follows here.

6.2 Theorem: *Let G be a commutative group; let S and V be GZ-related 4-element sets in G that do not include T-or-I related 3-element subsets [i.e., that are not of form (4.3)].*

Suppose that there are elements a and b in G such that both S and V can be expressed as a pair of elements that differ by ±a, and a disjoint pair of elements that differ by ±b. Then:

(A) a is of order 4, and

(B) there exists another element r, of order 4, such that the only common multiple of a and r in G is the 0-element, and

(C) there exist T-or-I operations OP and OP′ in G such that S = OP({0, a, r, −r − a}), while V = OP′({0, a, −r, r − a}).

PROOF: By the assumptions of the theorem, we may transpose or invert S and V so that, for some r and some u, S assumes the form {0, a, r, r + b}, while V assumes the form {0, a, u, u + b}. We shall suppose that S and V are already in those forms.

By the assumptions of the theorem, the interval-assortment of S matches, *en masse*, the interval-assortment of V. We can prune entries a, −a, b, and −b off both rosters; having done so, we conclude that

(i) [r, −r, r − a, a − r, r + b, −r − b, r + b − a, a − (r + b)] MEM [u, −u, u − a, a − u, u + b, −u − b, u + b − a, a − (u + b)].

Consequently, the element u, which appears on the pruned V-roster (right-side roster) of (i), must match (equal) something on the pruned S-roster (left-side roster) of (i).

6.2.1 We show that u must equal –r.

u cannot match r, because the sets S and V are assumed to have no common 3-element subset. For the same reason, u cannot match r + b. u cannot match a – r, for if it did, the a-inversion of S, {a, 0, a – r, a – (r + b)}, would be {a, 0, u, u – b},which would have 3 common elements with V. For the same reason, u cannot match a – (r + b). It follows, from (i), that

(ii) u must match something on the list [–r, r – a, –r – b, r + b – a]. Now

(iii) u cannot match –r – b.

For if it did, then u + b would equal –r, and the 0-inversion of set S, which is {–0, –a, –r, –(r + b)}, would be {0, –a, u + b, u}, so that the 0-inversion of S would have 3 common elements with V, contrary to supposition. Furthermore,

(iv) u cannot match r – a.

For if it did, then u + b would equal r + b – a, and the rosters of (i) above could be pruned as in (v) below:

(v) [r, –r, r + b, –r – b] matches, *en masse,* [u – a, a – u, u + b – a, a – (u + b)].

Substituting u = r – a (the assumption of (iv)) throughout the u-roster of (v), we would have

(vi) [r, –r, r + b, –r – b] MEM [r – 2a, 2a – r, r – 2a + b, 2a – r – b].

Now r, on the left of (vi), cannot match (equal) r – 2a, on the right. (That would entail 2a = 0, but by (5.2), S cannot span any interval of order 2.) Nor can r, on the left of (vi), match 2a – r on the right. (That would entail 2(a – r) = 0; S would then span the interval a – r of order 2.) r, on the left of (vi), in no way can match r – 2a + b on the right. For if it did, then we would have b = 2a. Furthermore, we could prune (vi) so that [r + b, –r – b] would match, *en masse,* [r – 2a, 2a – r]. Substituting b = 2a, we could infer that

(vii) [r + 2a, –r – 2a] MEM [r – 2a, 2a – r].

Thus either r + 2a = r – 2a, or r + 2a = 2a – r. If r + 2a = r – 2a, then a would be of order 4. But then b, which equals 2a, would be of order 2, and S would span an interval of order 2. That cannot happen. If on the other hand r + 2a = 2a – r, then r = –r and S would span the interval r of order 2. That cannot happen. So we have established (iv) above.

In light of (ii), (iii), and (iv) above, we infer

(ix) either u = –r, or u = r + b – a.

Exchanging the roles of S and V in the immediately preceding work, so that we exchange the roles of u and r symmetrically, we infer the symmetrically-related conclusion:

(x) either r = –u, or r = u + b – a.

It cannot be that *both* u = r + b – a (as in (ix)) *and* r = u + b – a (as in (x)), for in that case we would have b – a = 0, and b = a. (S does not span any interval in two different ways, via (5.4).) It follows that *either* u = –r (as in (ix)), *or* r = –u (as in (x)). And that is simply to say that

(xi) u = –r. (6.2.1) is thus accomplished.

Since u = –r, and –u = r, we can prune the rosters of (i) above, obtaining

(xii) [r – a, a – r, r + b, –r – b, r + b – a, a – (r + b)] MEM [u – a, a – u, u + b, –u – b, u + b – a, a – (u + b)].

Substituting u = –r, we get

(xiii) [r – a, a – r, r + b, –r – b, r + b–a, a – (r + b)] MEM [–r – a, a + r, b – r, r – b, –r + b – a, r + a – b].

Now r – a, on the left side of (xiii), does not match –r – a on the right. (r ≠ –r, since S contains no interval of order 2.) And r – a, on the left side of (xiii), does not match a + r on the right. (–a does not equal a.) Neither does r – a, on the left of (xiii), match r – b on the right. (–a ≠ –b, since S spans no interval in two different ways.) So

(xiv) r – a, on the left side of (xiii), matches something on the roster [b – r, –r + b – a, r + a – b].

6.2.2 We show that r – a must equal b – r.

The demonstration will take place via topics (xv)-(xvii), following.

(xv) r – a, in (xiv), cannot match –r + b – a.

For suppose it did. We could prune (xiii) to get

(xvi) [r + b, –r – b, r + b – a, a – (r + b)] matches, *en masse*, [–r – a, a + r, b – r, r – b].

Now, since r – a = –r + b – a, we infer that b = 2r. Substituting 2r for b in (xvi), we infer that

(xvii) [3r, –3r, 3r – a, a – 3r] MEM [–r – a, a + r, r, –r].

Then r, on the right-hand roster of (xvii), must match something on the left-hand roster there.

r cannot match 3r, since that would entail 2r = 0, and we know that S spans no interval of order 2.

If r matched –3r in (xvii), 3r would be –r, and r + b = 3r = –r = u. But then S would include the 3-element set {0, a, r + b} = {0, a, u}, and V would also include the same 3-element set. But we assumed that S and V had no common 3-element subset. So r, on the right side of (xvii), cannot match –3r on the left.

If r matched 3r – a in (xvii), then we would have a = 2r. But via (xv)—still in effect here—we have b = 2r. Thus a = b, which cannot happen. (S does not span any non-zero interval in two different ways.) So r does not match 3r – a in (xvii).

And r cannot match a – 3r in (xvii). For if it did, we would have a = 4r. Now (xvii) says that [3r, –3r, 3r – a, a – 3r] MEM [–r – a, a + r, r, –r]. Substituting 4r for a, we would obtain [3r, –3r, –r, r] MEM [–5r, 5r, r, –r]. Pruning, we would obtain [3r, –3r] MEM [–5r, 5r]. Thus 3r = 5r, or 3r = –5r. But if 3r = 5r, then 2r = 0 and S would span a non-zero interval of order 2. And if 3r = –5r, then 8r would equal 0; since a = 4r, we would have 2a = 0, and thus S would span a non-zero interval of order 2. None of this can happen. So r does not match a – 3r in (xvii).

The four preceding paragraphs have shown that r, on the right-hand roster of (xvii), cannot match anything on the left-hand roster of (xvii). Consequently (xvii) cannot happen. But (xvii) followed from (xv). So (xv) cannot happen.

We return then to (xiv), knowing that (xv) cannot be the case. Here is (xiv) again, for review:

(xiv) r – a, on the left side of (xiii), matches something on the roster [b – r, –r + b – a, r + a – b].

We must show that r – a = b – r, and we have just shown (in (xv)-(xvii)) that r – a cannot equal –r + b – a. Now we have to show that r – a cannot equal r + a – b, in the context of everything so far up through (xiii). We give (xiii) again, for convenient reference:

(xiii) [r – a, a – r, r + b, –r – b, r + b – a, a – (r + b)] MEM [–r – a, a + r, b – r, r – b, –r + b – a, r + a – b].

If r – a did match r + a – b, then we would have –a = a – b, and b = 2a. Pruning ±(r – a) off the left side of (xiii), and ±(r + a – b) off the right side, we would have

(xviii) [r + b, –r – b, r + b – a, a – (r +b)] matches, *en masse*, [–r – a, a + r, b – r, r – b].
Substituting b = 2a throughout (xviii), we would have

(xix) [r + 2a, –r – 2a, r + a, –r – a)] MEM [–r – a, a + r, 2a – r, r – 2a].
Pruning (xix), we would have

(xx) [r + 2a, –r – 2a] MEM [2a – r, r – 2a].
Consequently, r + 2a would have to equal either 2a – r or r – 2a. But neither of those things can happen. r + 2a cannot equal 2a – r, for if it did then r would equal -r, and S would span the interval r of order 2. And r + 2a cannot equal r - 2a, for if it did then 2a would be of order 2, so b = 2a would be of order 2; S would then span the interval b of order 2.

Since (xx) cannot happen, r – a, in (xiv), can thus match only b – r in (xiv). (6.2.2) is accomplished.

We give (xiii) again for convenient reference.

(xiii) [r – a, a – r, r + b, –r – b, r + b – a, a – (r + b)] MEM [–r – a, a + r, b – r, r – b, –r + b – a, r + a – b].

We know now that r – a = b – r, so we prune ±(r – a) off the left side of (xiii), and ±(b – r) off the right side, obtaining

(xxi) [r + b, –r – b, r + b – a, a – (r + b)] matches, *en masse*, [–r – a, a + r, –r + b – a, r + a – b].
Since r – a = b – r, we know that b = 2r – a. Substituting 2r – a for b throughout (xxi) above, we obtain

(xxii) [3r – a, a – 3r, 3r – 2a, 2a – 3r] MEM [–r – a, a + r, r – 2a, 2a – r].
Since a + r appears on the right side of (xxii), it must match something on the left side:

(xxiii) a + r matches something on the roster [3r – a, a – 3r, 3r – 2a, 2a – 3r].

6.2.3 We show that a + r, in (xxiii), must in fact equal a – 3r.

We proceed by eliminating the other three possibilities in (xxiii).

a + r cannot equal 3r – a, for if it did, then 2a would equal 2r, and a – r, an interval of S, would be of order 2.

a + r cannot equal 3r – 2a. For if it did, then 2r would equal 3a. Then b, which is 2r – a (from 6.2.2), would equal 3a – a. That is, we would have

(xxiv) b = 2a.

Pruning (xxii) above under the supposition of the present paragraph, we could remove ±(a + r) from the right side, and also ±(3r – 2a) from the left side. From this we would then infer that [3r – a, a – 3r] would have to match, *en masse*, [r – 2a, 2a – r]. But that cannot happen. For if 3r – a = r – 2a, then 2r would equal –a; whence (by the assumption of the present paragraph) 3a would equal –a; whence 4a = 0; whence 2b would equal 0 (xxiv), and S would span the interval b of order 2. Since [3r – a, a – 3r] has to match, *en masse*, [r – 2a, 2a – r], and since 3r – a (as just shown) cannot equal r – 2a, we would infer that 3r – a must equal 2a – r. But then 4r would equal 3a. However, the assumption of the present paragraph is that 2r = 3a. Thus 4r would equal 2r, and the interval r, spanned by S, would be of order 2.

a + r cannot equal 2a – 3r. For if it did, then we would infer a = 4r. Pruning (xxii) under the assumption of the present paragraph, we could remove ±(a + r) from the right-hand roster, and ±(2a – 3r) from the left, giving [3r – a, a – 3r] MEM [r – 2a, 2a – r]. But that cannot happen. For either Case 1 or Case 2 following would have to obtain: Case 1: 3r – a = r – 2a; Case 2: 3r – a = 2a – r. We show that neither of the two Cases can happen.

Case 1: We would have 3r – a = r – 2a, so that 2r = –a. Then 4r would equal –2a. But 4r would also equal a (second sentence of the paragraph above). We could then infer that 3a = 0, whence 2a = –a. Substituting the latter relation into the equation for this case, we could infer 3r – a = r – 2a = r + a. But then 2r would equal –2a, and 2(r – a) would be zero. So S would span the interval r – a of order 2, which cannot be so. Thus Case 1 cannot happen.

Case 2: From the equation 3r – a = 2a – r, we could infer 4r = 3a. But also 4r = a (as similarly assumed for Case 1). So 3a would be equal to a; a would be of order 2, and S would span an interval of order 2. So Case 2 cannot happen.

Since neither Case 1 nor Case 2 can happen, the supposition that [3r – a, a – 3r] MEM [r – 2a, 2a – r] cannot happen.

The preceding paragraphs have removed every alternative to the conclusion of (6.2.3). So (6.2.3) is established: a + r must equal a-3r.

6.2.4 The rest of the way to Theorem 6.2.

For convenient reference, we again give (xxii) above.

(xxii) [3r – a, a – 3r, 3r – 2a, 2a – 3r] MEM [–r – a, a + r, r – 2a, 2a – r].

Knowing now (6.2.3) that a + r = a – 3r, we can prune ±(a + r) from the right of (xxii), and ±(a – 3r) from the left. We obtain

(xxv) [3r – 2a, 2a – 3r] matches, *en masse*, [r – 2a, 2a – r].

Now in (xv), 3r – 2a cannot match r – 2a, since 2r is not 0. (S does not span an interval of order 2.) Hence, in (xxv), 3r – 2a must match 2a – r. So 4r = 4a. But also, via (6.2.3), 4r = 0. Therefore

(xxvi) Both a and r are of order 4.

(xxvi) establishes (A) of Theorem 6.2. To establish (B) of the theorem (since r is of order 4), we need to show that no non-zero multiple of a is also a multiple of r. That is, for integers j and k between 1 and 3 inclusive, we must show that ja + kr is not 0. We proceed to that demonstration by cycling through values of j and k.

a + r is not zero, since a and u = –r (by 6.2.1) are distinct members of V.

a + 2r is not zero. For r is of order 4, so 2r is of order 2. And a, an interval of S, is not of order 2.

a + 3r is not zero. For r is of order 4, so that 3r = –r. And a - r is not zero. (a and r are distinct members of S.)

2a + r is not zero. For a is of order 4 (by (xxvi)); consequently 2a is of order 2. And –r, an interval of S, is not of order 2.

2a + 2r is not zero. For a + r = a – u (6.2.1), and a – u is an interval of V. But then that interval of V is not of order 2. So 2(a + r) is not zero.

2a + 3r is not zero. For 3r is –r ((xxvi)), and we cannot have 2a – r = 0. We know this because r cannot be equal to 2a, because r is an interval of S, and 2a is of order 2 ((xxvi)).

3a + kr is not zero, for k = 1, 2, 3. For 3a + kr is the group-negative of a + (4 – k)r, and we have seen that a + (4 – k)r cannot be zero.

Accordingly, (B) of Theorem 6.2 is established.

Now at the beginning of our proof we transposed-or-inverted S and V to get them into the forms {0, a, r, r + b} and {0, a, u, u + b} respectively. Accordingly, (C) of Theorem 6.2 is established, and the entire Theorem is established. *Q.E.D.*

7. GZ-related tetrads, not of form (4.3), that cannot both be expressed as the union of an a-dyad and a b-dyad.

7.1 (5.1.2) above gave an example of a group G that contained 4-element GZ-related sets S and V, not of form (4.3), where it was *not* the case that for suitable intervals a and b, both S and V could be expressed as the disjoint union of an a-dyad and a b-dyad. This was noted at the end of the section discussing that example. The group in (5.1.2) was the additive group mod 13; the 4-element sets of the example, S = {0, 1, 4, 6} and V = {0, 2, 3, 7}, are GZ-related. If we transpose that V by 4 mod 13, the resulting two tetrads will still be GZ-related; they will be S = {0, 1, 4, 6} and V = {4, 6, 7, 11}.

In the work of section 7 following, we shall prove that all remaining GZ-tetrads in commutative groups are essentially of just this sort. More specifically, we shall prove

7.2 **Theorem:** *Let G be a commutative group; let S and V be GZ-related 4-element sets in G that do not include T-or-I related 3-element subsets [i.e., that are not of form (4.3)].*

Suppose that there are not *any elements a and b in G such that both S and V can be expressed as a pair of elements that differ by ±a, and a disjoint pair of elements that differ by ±b.*

Then there exists an element x of G which is of order 13, and there exist T-or-I operations OP and OP′, such that

either

S = OP({0, x, 4x, 6x}) and V = OP′({4x, 6x, 7x, 11x}),

or

V = OP({0, x, 4x, 6x}) and S = OP′({4x, 6x, 7x, 11x}).

PROOF: Transposing or inverting S and V as needed, we can suppose that S = {0, a, r, r + b}, while V = {r, r + b, u, u + c}, where the six elements a, –a, b, –b, c, and –c are all distinct. (By the supposition of the theorem, if S is the union of an a-dyad and a b-dyad, then V is not the union of a b-dyad and an a-dyad. The twelve non-zero intervals of S are all distinct (by (5.4)), as are the twelve non-zero intervals of V.)

Since c is an interval of V, it is an interval of the GZ-related S. Since c is distinct from ±a, and from ±b, spanning c within S = {0, a, r, r + b} must involve either 0 or a, and either r or r + b.

7.2.1 We may suppose that spanning c within S involves r, but not r + b.

To see this, let us suppose that spanning c within S involves r + b. Set r′ = r + b and b′ = –b. Then we have S = {0, a, r′, r′ + b′}, while V = {r′, r′ + b′, u, u + c}, and spanning c within S involves r′. So, by transforming the labels for elements of our GZ-related tetrads, we can make the supposition of (7.2.1).

7.2.2 We may suppose that spanning c within S involves 0, not a.

To see this, let us suppose that spanning c within S involves a. Then, via (7.2.1), either c = a – r, or c = r – a.

Case 1: c = a – r. Now S = {0, a, r, r + b}, so the a-inversion of S, which we shall call S′, is {a, 0, a – r, a – r – b} = {a, 0, c, c – b} by the assumption of Case 1. Set b′ = –b; then S′ = {0, a, c, c + b′}.

And V = {r, r + b, u, u + c} = {r, r – b′, u, u + c}, so the b′-transposition of V, which we shall call V′, then becomes {r + b′, r, u + b′, u + b′ + c}. Set u′ = u + b′; and thus V′ = {r, r + b′, u′, u′ + c}.

Then the (c – r)-transposition of V′, which we shall call V″, is {c, c + b′, c – r + u′, c – r + u′ + c}. Set u″ = c – r + u′; then V″ = {c, c + b′, u″, u″ + c}

In sum, transposing, inverting, and shuffling labels has allowed us to express our GZ-related tetrads as forms of S′ = {0, a, c, c + b′} and V″ = {c, c + b′, u″, u″ + c}. Spanning c within S′ involves 0, not a. So in Case 1 here, (7.2.2) is established.

Case 2: c = r – a. Now S = {0, a, r, r + b}, and the ensuing (–a)-transposition of S, which we call S′, is {–a, 0, r – a, r – a + b}, which is {–a, 0, c, c + b} by the assumption of this case. Set a′ = –a; then S′ = {0, a′, c, c + b}.

And V = {r, r + b, u, u + c} = {a + c, a + c + b, u, u + c} (as assumed for this case), so the (–a)-transposition of V, which we shall call V′, is {c, c + b, u – a, u – a + c}. Set u′ = u – a; then we can express V as {c, c + b, u′, u′ + c}.

In sum, transposing, inverting, and shuffling labels has allowed us to express our GZ-related tetrads as forms of S′ = {0, a′, c, c + b} and V′ = {c, c + b, u′, u′ + c}. Spanning c within S′ involves 0, not a′. So in Case 2 here, (7.2.2) is established. Since Cases 1 and 2 are exhaustive, (7.2.2) is established.

Thus we may suppose that spanning interval c within S involves r, not r + b ((7.2.1)), and we may also suppose that spanning interval c within S involves 0, not a ((7.2.2)). Thus we may suppose that either c = r (= r – 0), or c = –r (= 0 – r).

7.2.3 We may suppose that c = r (and not –r).

If c = –r, we set c′ = –c, and u′ = u + c. Then S = {0, a, r, r + b} = {0, a, –c, –c + b} = {0, a, c′, c′ + b}, while V = {r, r + b, u, u + c} = {–c, –c + b, (u + c) – c, u + c} = {c′, c′ + b, u′ + c′, u′}. By shuffling labels, we thus express our GZ-related tetrads as forms of S = {0, a, c′, c′ + b} and V = {c′, c′ + b, u′, u′ + c′}, in which c′ = r. (7.2.3) is established.

7.2.4 Summing up our work so far, we may suppose that S = {0, a, c, c + b}, while V = {c, c + b, u, u + c}, where ±a, ±b, and ±c are all distinct.

Now V spans interval a somehow, using either c or c + b, and either u or u + c.

7.2.5 We may suppose that u, and not u + c, is involved in spanning interval a within V.

Demonstration: Let V′ be the (2c + b)-inversion of V. Then V′ = {2c + b – c, 2c + b – (c + b), 2c + b – u, 2c + b – (u + c)} = {c + b, c, 2c + b – u, c + b – u}. Set u′ = c + b – u. Then set V′ = {c + b, c, u′ + c, u′}.

If (u + c) is involved in spanning a within V, then we have the (2c + b)-inversion of (u + c) involved in spanning –a within V′—that is, (2c + b) – (u + c), which is c + b – u, which is u′, is involved in spanning –a within V′. Since u′ is involved in spanning –a, u′ is also involved in spanning a within V′. By using V′ instead of V, as a model form for one of our GZ-tetrads, we establish (7.2.5).

Let us sum up so far.

7.2.6 We can express our GZ-tetrads as T-or-I forms of S = {0, a, c, c + b} and V = {c, c + b, u, u + c}, and we may suppose that u, not u + c, is involved in spanning interval a within V. We may then distinguish four possible cases. Case 1: a = u – c. Case 2: a = c – u. Case 3: a = c + b – u. Case 4: a = u – (c + b).

7.2.7 In Case 1 of (7.2.6), the conclusion of the theorem obtains.

PROOF: Given a = u – c, then we may substitute a + c for u in the expression of V. Then S = {0, a, c, c + b}, and also V = {c, c + b, a + c, a + 2c}. The interval-roster of S is [a, –a, c, –c, c – a, a – c, c + b, –c – b, c + b – a, a – c – b, b, –b], and we see the *en-masse* matching roster of V (as above) is [b, –b, a, –a, a – b, b – a, a + c, –a – c, a + c – b, b – a – c, c, –c]. We prune the rosters of ±a, ±b, and ±c; then

(i) [c – a, a – c, c + b, –c – b, c + b – a, a – c – b] matches, *en masse*, [a – b, b – a, a + c, –a – c, a + c – b, b – a – c].

Now c – a, on the left roster of (i) above, cannot match b – a on the right. (c ≠ b.) Nor can c – a (on the left) match a + c on the right. (a ≠ –a.) Nor can c – a (on the left) match –a – c on the right. (c ≠ –c.) Consequently

(ii) c – a, on the left of (i), must match a – b, or a + c – b, or b – a – c, on the right.

Now c – a, in (ii), cannot match a – b. For if the two were the same, then we would have c + b = 2a. In that case S, which is {0, a, c, c + b} ((7.2.6)), would equal {0, a, c, 2a}. S would then span the interval a in two different ways, as a – 0, and as 2a – a. But that is impossible, via (5.4). And c – a, in (ii), cannot match a + c – b. For if the two were the same, then we would have b = 2a. In that case V, which equals {c, c +b, u, u + c} ((7.2.6)), would become {c, c + 2a, c + a, 2c + a} (because, in the present assumption of Case 1 of (7.2.6), u = c + a). V would then span the interval a in two different ways, as (c + a) – c, and as (c + 2a) – (c + a). But that is impossible, via (5.4). As a result of this paragraph and the last, and (ii) above,

(iii) we must have c – a = b – a – c. Hence b = 2c.

From (i) above, we know that the roster [c – a, a – c, c + b, –c – b, c + – a, a – c – b] matched, *en masse*, the list [a – b, b – a, a + c, –a

$- c, a + c - b, b - a - c]$. Via (iii), we can prune $\pm(c - a)$ from the first of these rosters, and $\pm(b - a - c)$ from the second. We obtain

(iv) $[c + b, -c - b, c + b - a, a - c - b]$ MEM $[a - b, b - a, a + c, -a - c]$.

Using (iii), we can substitute $b = 2c$ into (iv). We obtain

(v) $[3c, -3c, 3c - a, a - 3c]$ MEM $[a - 2c, 2c - a, a + c, -a - c]$.

Now $a + c$, on the right of (v), must match something on the left of (v). We see that:

$a + c$ cannot match $3c$ on the left, for in that case we would have $a = 2c$. But b also equals 2c (via (iii) above), so S would span the interval $a = b$ in two different ways, contradicting (5.4).

$a + c$ cannot match $3c - a$ on the left, for in that case we would infer $2a = 2c$, and the interval $c - a$ of S would be of order 2, contradicting (5.2).

$a + c$ cannot match $a - 3c$ on the left, for in that case we would infer $4c = 0$. Since $b = 2c$ (via (iii)), 2b would equal 0, and S would span the interval b of order 2, contrary to (5.2).

The only remaining possibility, from (v), is that

(vi) $a + c = -3c$. Hence $a = -4c$.

Now (v) tells us that the roster $[3c, -3c, 3c - a, a - 3c]$ matches, *en masse*, the roster $[a - 2c, 2c - a, a + c, -a - c]$. Using (vi) to prune away $\pm(-3c)$ on the left roster, and $\pm(a + c)$ on the right, we infer that the roster $[3c - a, a - 3c]$ MEM $[a - 2c, 2c - a]$. Substituting $a = -4c$ (from (vi)), we infer that the roster $[7c, -7c]$ matches, *en masse*, the roster $[-6c, 6c]$. Now we cannot have $7c = 6c$. (c is not 0.) So we must have $7c = -6c$. And hence

(viii) $13c = 0$; the element c is of order 13.

Now we can push all the way to the desired conclusion. Because we are assuming Case 1 of (7.2.6), we can express the tetrads $S = \{0, a, c, c + b\}$ and $V = \{c, c + b, u, u + c\}$ as $S = \{0, a, c, c + b\}$ and $V = \{c, c + b, a + c, a + 2c\}$. (Case 1 supposes that $u = a + c$.) Because of (iii) and (vi) above, we can substitute 2c for b, and $-4c$ for a; the tetrads S and V are then expressed as $S = \{0, 4c, c, 3c\}$ and $V = \{c, 3c, -3c, -2c\}$. The 7c-inversion of S is then $\{7c, 11c, 6c, 4c\}$, while the 3c-transposition of V is $\{4c, 6c, 0, c\}$. Thus some OP-forms of V and S are respectively $\{0, c, 4c, 6c\}$ and $\{4c, 6c, 7c, 11c\}$.

Taking x = c, we see that the OP-forms of V and S are in the form demanded by Theorem 7.2. The theorem obtains in Case 1, as asserted.

We review where we are, in the proof of the theorem.

7.2.8 We can express our GZ-tetrads as T-or-I forms of S = {0, a, c, c + b} and V = {c, c + b, u, u + c}, and we may suppose that u, not u + c, is involved in spanning interval a within V. We distinguished four possibilities. Case 1: a = u – c; Case 2: a = c – u; Case 3: a = c + b – u; and Case 4: a = u – (c + b). We have shown that in Case 1 the conclusion of our theorem obtains.

7.2.9 Case 2 of (7.2.8) cannot happen.

PROOF: Assuming a = c – u, we can write u = c – a. Then we can express our GZ-tetrads S and V as S = {0, a, c, c + b} and V = {c, c + b, c – a, 2c – a}. The interval-roster for S is [a, –a, c, –c, c – a, a – c, c + b, –c – b, c + b – a, a – c – b, b, –b]; for V, [b, –b, –a, a, a + b, –a – b, c – a, a – c, c – b – a, a + b – c, c, –c]. We can prune those rosters of ±a, ±b, ±c, and ±(c – a). We obtain

(i) [c + b, –c – b, c + b – a, a – c – b] MEM [a + b, –a – b, c – b – a, a + b – c].

Now c + b, on the left roster of (i), must match something on the right roster. We proceed as before:

c + b cannot match a + b (on the right), since c ≠ a.

c + b cannot match c – b – a (on the right), for that would entail a = –2b. c – a would then equal c + 2b, and V, which is {c, c + b, c – a, 2c – a}, would become {c, c + b, c + 2b, 2c + 2b}. V would then span the interval b in two different ways: as (c + b) – c, and as (c + 2b) – (c + b). But that would contradict (5.2).

c + b cannot match a + b – c (on the right), for that would entail a = 2c. Then S, which is {0, a, c, c + b}, would equal {0, 2c, c, c + b}, and S would span the interval c in two different ways: as c – 0 and as 2c – c. But that would contradict (5.2). We conclude, from this paragraph and the two preceding, that

(ii) c + b must equal –a – b. Hence c – a, a non-zero interval of S, equals –2(a + b).

We prune (i) above, deleting ±(c + b) on the left, and ±(–a – b) on the right. We get

(iii) [c + b – a, a – c – b] MEM [c – b – a, a + b – c].
We substitute c = –a – 2b, from (ii), into (iii), and we get

(iv) [–2a – b, 2a + b] MEM [–2a – 3b, 2a + 3b].

Now 2a + b, on the left of (iv), cannot equal 2a + 3b, on the right. (b cannot equal 3b, for b is not of order 2, via (5.2).) So 2a + b, on the left of (iv), must equal –2a – 3b, on the right. So we have

(v) 4a = –4b; 4(a + b) = 0.

Let us set j = 2(a + b). (ii) above tells us that j is a non-zero interval of S. (v) above tells us that 2j = 0. So j is of order 2. (5.2) is contradicted. Thus Case 2 cannot happen, as asserted in (7.2.9).

We review where we are, in the proof of the theorem.

7.2.10 We can express our GZ-tetrads as T-or-I forms of S = {0, a, c, c + b} and V = {c, c + b, u, u + c}, and we may suppose that u, not u + c, is involved in spanning interval a within V. We have named four possible cases. Case 1: a = u – c. Case 2: a = c – u. Case 3: a = c + b – u. Case 4: a = u – (c + b). We have shown that in Case 1 the conclusion of our theorem obtains, and that Case 2 cannot happen.

7.2.11 Case 3 cannot happen.

Under the assumption of Case 3 (that a = c + b – u), we have u = c + b – a. Consequently we can express our GZ-tetrads as S = {0, a, c, c + b} and V = {c, c + b, u, u + c} = {c, c + b, c + b – a, 2c + b – a}. S's interval-roster is [a, –a, c, –c, c – a, a – c, c + b, –c – b, c + b – a, a – c – b, b, –b], and the interval-roster of V is [b, –b, b – a, a – b, –a, a, c + b – a, a – c – b, c – a, a – c, c, –c]. Pruning away ±a, ±b, ±c, ±(c – a), and ±(c + b – a) from both rosters, we get

(i) [c + b, –c – b] MEM [b – a, a – b].

Now c + b, on the left of (i), cannot match b – a on the right. Since S = {0, a, c, c + b}, c cannot equal –a; if c did equal –a, then S would span the interval –a in two different ways, as 0 – a, and as c – 0. That would contradict (5.4).

And c + b, on the left of (i), cannot match a – b on the right. That would entail c = a – 2b; then S, which is {0, a, c, c + b}, would equal {0, a, a – 2b, a – b}, and S would span the interval b in two different ways, as a – (a – b), and as (a – b) – (a – 2b). That would contradict (5.4).

So (i) above is impossible, and Case 3 indeed cannot happen. We review where we are now, in the proof of our theorem.

7.2.12 We can express our GZ-tetrads as T-or-I forms of S = {0, a, c, c + b} and V = {c, c + b, u, u + c}, and we may suppose that u, not u + c, is involved in spanning interval a within V. Four possible cases were distinguished. Case 1: a = u – c. Case 2: a = c – u. Case 3: a = c + b – u. Case 4: a = u – (c + b). We have shown that in Case 1 the conclusion of our theorem obtains, and that Case 2 cannot happen, and that Case 3 cannot happen.

7.2.13 In Case 4, the conclusion of the theorem obtains.

We could apply a lot of transformations to put Case 4 into the format of Case 1, but I think it will be useful for a reader to go over Case 4 as it stands in (7.2.12).

Under the assumption of Case 4 (that a = u – (c + b)), we have u = a + b + c. Consequently we can express our GZ-tetrads as S = {0, a, c, c + b} and V = {c, c + b, u, u + c} = {c, c + b, a + b + c, a + b + 2c}. We obtain the interval-roster for S: [a, –a, c, –c, c – a, a – c, c + b, –c – b, c + b – a, a – c – b, b, –b]. V's interval-roster is [b, –b, a + b, –a – b, a, –a, a + b + c, –a – b – c, a + c, –a – c, c, –c]. Pruning away ±a, ±b, and ±c from both rosters, we get

(i) [c – a, a – c, c + b, –c – b, c + b – a, a – c – b] matches, *en masse,* [a + b, –a – b, a + b + c, –a – b – c, a + c, –a – c].

Now c – a, on the left roster of (i), must match something on the right roster. c – a cannot match –a – b, for c ≠ -b. (S cannot span interval c in two different ways.) And c – a cannot match c + a, for –a is not equal to a. (S spans no interval of order 2.) And c – a cannot match –a – c, for c does not equal –c. (S spans no interval of order 2.) We then conclude

(ii) c – a, on the left side of (i), either equals a + b on the right, a + b + c on the right, or –a – b – c on the right.

Now c – a cannot equal a + b + c. For if that were the case, then V, which is {c, c + b, a + b + c, a + b + 2c} (under the assumption of Case 4), would equal {c, c + b, c – a, 2c – a}. But then V would span the interval a as c – (c – a), and V would span the interval a in another way, as (c – a) – (c + b), or (a + b + c) – (c + b). However, V does not span the interval a in two different ways.

Furthermore, $c - a$ cannot equal $-a - b - c$. For if that were the case, we could infer $c + b = -c$; then $S = \{0, a, c, c + b\}$ would equal $\{0, a, c, -c\}$, and S would span the interval c in two different ways, as $c - 0$, and as $0 - (-c)$.

The above two paragraphs, along with (ii) above, enable us to conclude that

(iii) $c - a$ must equal $a + b$.

We can then prune (i) above, removing $\pm(c - a)$ on the left roster, and $\pm(a + b)$ on the right. We obtain

(iv) $[c + b, -c - b, c + b - a, a - c - b]$ matches, *en masse*, $[a + b + c, -a - b - c, a + c, -a - c]$.

From (iii) we infer that $c = 2a + b$. Substituting $2a + b$ for c throughout (iv), we have

(v) $[2a + 2b, -2a - 2b, a + 2b, -a - 2b]$ MEM $[3a + 2b, -3a - 2b, 3a + b, -3a - b]$.

Now $a + 2b$, on the left of (v), must match something on the right of (v).

$a + 2b$ cannot match $3a + 2b$ (for that would entail $0 = 2a$, and a cannot be of order 2).

$a + 2b$ cannot match $-3a - 2b$ on the right. For that would entail $4(a + b) = 0$. Set $j = 2(a + b)$; then j would be of order 2. But j is an interval of S. (j appears on the left roster of (v).) This cannot happen.

$a + 2b$ cannot match $-3a - b$ on the right. For that would entail $4a + 3b = 0$, whence $2(2a + b) + b = 0$, whence (via (iii) above) $2c + b = 0$. But then $V = \{c, c + b, a + b + c, a + b + 2c\}$ (7.2.13, paragraph 2), would be $\{c, c + b, a + b + c, a\}$, and V would have elements c, $c + b$, and a in common with set $S = \{0, a, c, c + b\}$. But S and V do not have three common elements (by the assumption of Theorem 7.2). So this cannot happen. In light of this paragraph, plus the two preceding paragraphs, we conclude that

(vi) $a + 2b$ must equal $3a + b$, whence $b = 2a$.

From (iii) above, we infer that $c = 2a + b$. So $c = 2b = 4a$.

The second paragraph of (7.2.13) gives our GZ-tetrads, in the present case (Case 4), as set $S = \{0, a, c, c + b\}$ and $V = \{c, c + b, a + b + c, a + b + 2c\}$. Substituting $b = 2a$ and $c = 4a$ throughout, we can express the tetrads as $S = \{0, a, 4a, 6a\}$ and $V = \{4a, 6a, 7a, 11a\}$.

And, taking x = a, that is exactly a form demanded by Theorem 7.2, which theorem is therefore satisfied in Case 4.

We have finished: Theorem 7.2 is true in Case 1 (Step 7.2.7); the theorem is also true in Case 4 (7.2.13); Cases 2 and 3 cannot happen (7.2.9 and 7.2.11).

8. A neatly organized summary of all the above.

8.1 Let G be a commutative group (any commutative group whatsoever). Let S and V be GZ-related tetrads of elements from G. We define S and V to be of Category 1 when they can be transformed by T-or-I operations so that they have three members in common. We define S and V to be of Category 2 when they are not of Category 1, and when there exist group-elements a and b so that S can be expressed as the union of an a-dyad with a b-dyad, and V can also be expressed as the union of an a-dyad with a b-dyad. We define S and V to be of Category 3 when they are not of Category 1, and not of Category 2.

The three Categories, as defined just above, are exhaustive: GZ-related tetrads S and V must be in one (and only one) of the three Categories.

8.2 We have proved (in (3.3) and (4.2)) that S and V are of Category 1 if, and only if, there exist group elements g and k, and T-or-I operations OP and OP′, such that g is of order 4, k is not a multiple of g, k is not of order 2, 2k is not equal to 2g, S = OP({0, 2g, k – g, k}). and V = OP′ ({0, 2g, k, k + g}).

8.3 We have proved (in (6.2)) that S and V are of Category 2 if, and only if, there exists an element a of order 4, and another element r of order 4, such that the only common multiple of a and r is the 0-element, and there exist T-or-I operations OP and OP′, such that S = OP({0, a, r, –r – a}), while V = OP′ ({0, a, –r, r – a}).

8.4 We have proved (in (7.2)) that S and V are of Category 3 if, and only if, there exists an element x of order 13, and T-or-I operations OP and OP′, such that either S = OP({0, x, 4x, 6x}) and

V = OP′ ({4x, 6x, 7x, 11x}), or else V = OP({0, x, 4x, 6x}) and S = OP′ ({4x, 6x, 7x, 11x}).

8.5 Along the way, we observed (in (5.4)) that if S and V are not of Category 1, then S can span no non-zero interval in two different ways. In particular then (as per (5.2)), S has no interval of order 2.