Aspects of Recursion in M-Inclusive Networks

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1. Introduction

We examine transformational schemes that incorporate M_n operations,[1] with the intention of building recursive networks-of-networks.[2] In the process, we discuss K-net isographies, Lewin's RECURSE group,[3] and isographies that derive from inner and outer automorphisms[4] of the T/M group,[5] all of which present certain difficulties in realizing our goal. Ultimately, we define a group of graphic transformations that is isomorphic to T/M. This group allows us to model pitch-class networks recursively as networks-of-networks, and so on.

M_n and MI_n operations both exchange interval-classes (ics) 1 and 5. For this reason, they are particularly germane in transformational analysis of music that incorporates interactions between diatonic spaces—or, more generally, the circle-of-fourths/fifths spaces of which diatonic spaces are connected

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[1] Throughout this study, we use the notation M_n and MI_n for multiplicative operations. Specifically, M_n obtains from multiplication by 5, followed by transposition by n (modulo 12). MI_n uses multiplication by 7, followed by transposition by n (modulo 12).

[2] Networks-of-networks(-of-networks, etc.) were first described in Lewin 1990, 93-97. See also Lambert 2002, 181-82.

[3] Lewin 1990, 117-20.

[4] The Glossary at the end of this article contains definitions of mathematical terms—such as inner and outer automorphisms—which are not encountered commonly in music-theoretical discourse.

[5] We use the notation T/M to signify the group of T_n and M_n operators (similar to the common practice of using T/I to represent the group of T_n and I_n operators). The reader is cautioned not to confuse this notation with Lewin 1990, 117, where "T/M" is the group of forty-eight twelve-tone operators (including the T_ns, I_ns, M_ns, and MI_ns). We use the notation TTO_{48} for the latter, following Morris 1987.

segments[6]—and chromatic spaces. The groups T/M and T/MI alone, but not together,[7] are also especially relevant in the analysis of music that does not incorporate inversion structurally.

Morris discusses an important feature of the twelve-tone operators (TTOs)[8] (Appendix, Table A.1): T_n and I_n operations preserve *distances* among pitch-classes (pcs) in pc-space; as such, they also preserve the *ratios* of those distances. M_n and MI_n operations, on the other hand, preserve the ratios, but not necessarily the distances. For example, pcset {0,1,3}, a member of set-class (sc) 3-2[013], contains intervals that belong to ics 1, 2, and 3. The image of this pcset under any T_n or I_n operator is also a member of 3-2[013], and preserves the distances of those intervals in pc space. Clearly, these operations also preserve the ratios of those distances (i.e., intervals from ic 1 are 1/12 of an octave in pc space, those from ic 2 are 1/6, and those from ic 3 are 1/4). In contrast, the image of {0,1,3} under either M_n or MI_n is a member of sc 3-7[025], whose members contain intervals that belong to ics 2, 3, and 5. Whereas the distances of the intervals from ics 2 and 3 are preserved, the interval from ic 1 maps to an interval from ic 5, not preserving distance. Nevertheless, the ratios of all intervals are indeed preserved; in particular, we note that intervals from both ics 1 and 5 are 1/12th of an octave in pc space.[9] In hearing M_n and MI_n operators, we must concentrate on these ratios, and not on specific distances.

2. Toward M- and MI-inclusive Networks

In transformation theory that incorporates networks, such networks must be well-formed; that is, their various transformational pathways must be equal.[10] For instance, any two

[6] Clough, Engebretsen, and Kochavi 1999, 76 list the usual diatonic as an example of a generated set. As such it is (a segment of) a cyclic set that is generated by a single interval (from ic 5).

[7] The product of any two M_n and MI_n operators is some I_n.

[8] Morris 2001, 52.

[9] See Peck 2002, 159-61 for a related discussion.

[10] O'Donnell 1998, 57 discusses further the concept of well-formedness in K-nets.

edges of a trichordal graph must combine appropriately to equal the third. Using an L-net,[11] Example 1a interprets the pcset {9,11,0}. Here, C moves to B via T_{11}, and B moves to A via T_{10}. These two operators, T_{11} and T_{10}, must combine to equal the same operator, T_9, which moves C to A; in fact, $(T_{10})(T_{11}) = T_9$, as $T_xT_y = T_{x+y}$.[12] The remaining figures in Examples 1 display all the other possible L-net interpretations of the set.[13]

The well-formedness condition must also hold if we use other types of operators in addition to T_n. In K-net theory, for instance, any trichordal graph must contain one T_n arrow and two I_n arrows.[14] Its two I_n operators combine then to form its T_n operator, as $I_xI_y = T_{x-y}$ (and $I_yI_x = T_{-(x-y)}$). Furthermore, its T_n operator and either of its I_n operators form together the remaining I_n operator, as $T_{x-y}I_y = I_yT_{-(x-y)} = I_x$ and $I_xT_{x-y} = T_{-(x-y)}I_x = I_y$. Example 2 reinterprets the above trichord, showing the six possible combinations of T_n and I_n arrows.[15] These six networks divide into three pairs (Examples 2a and b, c and d, and e and f) whose members have the same Generalized Interval System (GIS) content.[16] Together with Example 1, this figure suggests the

[11] O'Donnell 1998, 53 labels T_n-only networks, such as one finds in Lewin 1987, as "L-nets," for "Lewin networks." He then uses "K-nets" for "Klumpenhouwer networks," which incorporate both T_n and I_n operators. Henceforth we will use these abbreviations.

[12] We use left-functional orthography in this study, following the standard practice for compositions of TTOs in the music-theoretical literature. That is, the composition *ab* means: "do *b* first, then do *a*."

[13] For a trichordal L-net, wherein each of the three arrows may move either of two directions, we find $2^3 = 8$ possible interpretations.

[14] O'Donnell 1998, 57.

[15] For a T_n and I_n trichordal network, we find only six possible interpretations. To preserve well-formedness, only one of the three edges may be represented by a T_n arrow. The remaining I_n arrows represent involutions; as such, we cannot distinguish their direction (and we give them typically as double arrows). Consequently, as the T_n arrow may represent one of three edges, and may move in either of two directions, we find $3 \cdot 2 = 6$ well-formed interpretations.

[16] Each pair contains graphs that associate the same edge with a T_n arrow, with one T_n arrow that goes one direction and the other the opposite. Then, the members of these pairs possess the same total GIS interval content. Recall that for a GIS (S, IVLS, int), int is a function that maps the Cartesian product of a set, $S \times S$, to a group IVLS. Therefore, as (a,b) and (b,a) are members of $S \times S$, for any a,b

considerably large number of interpretations of a pcset available to an analyst who uses transformational networks. Indeed, the choice of network—that is, the interpretation—is an important part of the process of network analysis.[17]

We may extend the concept of well-formedness to graphs that incorporate M_n and MI_n operators as well (the rules of combination are listed in Table A.2). Example 3 shows various further interpretations of the above trichord, all of which are well-formed. Reference to the cycles of the M_n and MI_n operators acts as an aid (Table A.1). We note the T_{11}, M_9, and M_2 operators in Example 3a. The cycle of M_9 from pc 0 is (0, 9, 6, 3). The cycle of M_2 from pc 9 is (9, 11). The latter is an involution, so the M_2 arrow in Example 3a is double-headed, the same as an I_n operator's. In combination, $M_2M_9 = T_{2+(5\cdot 9)} = T_{11}$, whose cycle from 0 is (0, 11, 10, ..., 1). In the same way, we can examine the remaining graphs in the example, and this list is certainly not exhaustive. Furthermore, in addition to trichordal graphs, the concept of well-formedness extends to graphs of any size.

Whereas recursion need not be the ultimate goal in a transformational theory, its functioning in music analysis—for example, in certain aspects of Schenkerian analysis, or in Schoenberg's concept of *Grundgestalt*—is often telling of the relationship between larger- and smaller-scale structures in a piece of music. For that reason, we seek a system of recursive modeling for our M-inclusive networks. In network analysis in general, recursion works in the following way: a set of networks represents a set of interpretations, and these interpretations may share some degree of structural identification.[18] On a subsequent hierarchic

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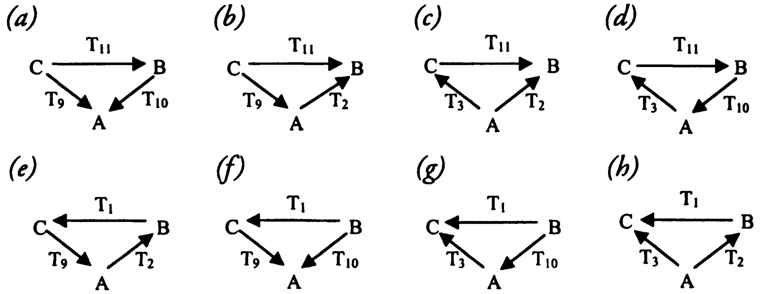
$\in$ S, the GIS interval content—that is, the images of the members of S × S in IVLS—of any graph contains its labeled operators *as well as their inverses*. See Lewin 1987, 24-30 for further discussion of GISs.

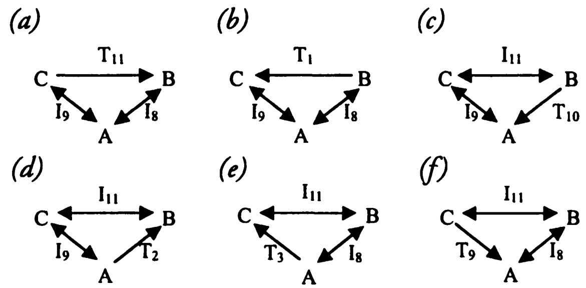
[17] Lewin 1990 and 1994 deals significantly with the issues of choosing appropriate interpretations in K-net analysis.

[18] Indeed, network analysts often choose network interpretations to reflect such associations.

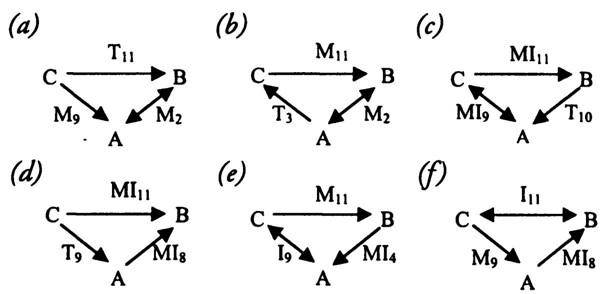
Example 1. The eight possible L-nets of {9,11,0}.



Example 2. The six possible K-net interpretations of {9,11,0}.



Example 3. Some further well-formed interpretations of {9,11,0}.



level, one may relate these interpretations in terms of this identification; that is, the analyst interprets the networks themselves in a particular way. Again, structural identifications may exist between this process of interpretation and those of the individual networks, and such conformity results in analytical recursion.[19] The types of relationships that may exist between the two levels vary, but in network analysis they include most often the same number of interpretative acts, corresponding lengths of the cycles of these interpretations, and corresponding results of combinations of various interpretive acts.[20]

In considering hyper-operators that share a structural identification with the M_n operators, we recall our previous discussion of distance and ratio. Hyper-M_n operators do not necessarily need to preserve the specific *distances* among interpretations within cycles, but they need to preserve the *ratios* of these distances. Again, in hearing hyper-M_n operations, one must focus on these ratios.

3. Isography and <u, j, p>-nets

In K-net theory, analysts describe such recursive relationships as graph isomorphisms and isographies.[21] Lewin defines isography between two networks in terms of the following three features: "(1) They must have the same configuration of nodes and arrows. (2)

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[19] For a related discussion, see Lewin 1990, 93-97.

[20] Whereas "the same number of interpretative acts" is self-explanatory, "the lengths of the cycles of these interpretations" and "results of combinations of various interpretive acts" may not be so. We may interpret network A in terms that relate it to network B, then interpret B in precisely the same way, relating it to C. The length of the cycle of interpretations is the number of the same interpretive acts it would take ultimately to relate some subsequent network N to A. If we interpret network X in terms that relate it to network Y, then interpret Y in terms (not necessarily the same) that relate it to Z, we have a product of interpretations from X to Z. This product is the result of combinations of various interpretive acts.

[21] Graph isomorphisms relate graphs without defined node content. Once we define the contents of the nodes, the graphs become networks, which we relate by isographies.

There must be some isomorphism F that maps the transformation-system used to label the arrows of one network, into the transformation-system used to label the arrows of the other. (3) If the transformation X labels an arrow of the one network, then the transformation F(X) labels the corresponding arrow of the other."[22]

The crux of the concept of isography lies in the isomorphisms that map one transformation-system to others. Among networks that use the same group G of operators, these isomorphisms derive from the group of automorphisms of G, AUT(G). For the T/I group used traditionally in K-net analysis, AUT(T/I) contains the forty-eight unique mappings of T/I onto itself that satisfy the above condition. Lewin defines its forty-eight mappings in terms of an ordered duple <u, j>, where u = 1, 5, 7, or 11, and j is an integer modulo 12.[23] This duple permits distinct actions on the T_n and I_n operators. Specifically, the image of any T_n operator under <u, j> is T_{un}, and the image of any I_n operator is I_{un+j}. Accordingly, the graph of a K-net—that is, in terms of its labeled arrows, and without regard to node content—has potentially forty-eight isomorphic counterparts, including itself. In fact, AUT(T/I) is isomorphic to TTO_{48}.[24]

Whereas forty-eight automorphisms of the T/I group exist, we find ninety-six automorphisms of TTO_{48}. Therefore, graphs that incorporate M_n and MI_n operators, in addition to T_ns and I_ns, have potentially ninety-six isomorphic counterparts. Lewin gives these automorphisms in the form of an ordered triple <u, j, p>, where u = 1, 5, 7, or 11, j is an even integer modulo 12; and p is a multiple of 3 modulo 12.[25] Lewin's reason for using this triple, rather than the above duple, is that it permits distinct actions on the T_n, M_n, MI_n, and I_n operators as follows: $\langle u, j, p\rangle T_n = T_{un}$, $\langle u, j, p\rangle M_n = M_{un+j}$, $\langle u, j, p\rangle MI_n = MI_{un+p}$, and $\langle u, j, p\rangle I_n = I_{un+j+p}$.[26]

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[22] Lewin 1990, 87.

[23] *Ibid.*, Appendix A.

[24] The isomorphism of AUT(T/I) to TTO_{48} can be seen as follows. TTO_{48} may be generated by T_1, M_0, and MI_0. Then, AUT(T/I) may be generated similarly by <1,1>, <5,0>, and <7,0>, which have the same structure as T_1, M_0, and MI_0.

[25] Lewin 1990, 118-19.

[26] We note that AUT(TTO_{48}) is isomorphic to the abstract direct product group $D_6 \times D_8 \times C_2$ (where D_n is the dihedral group of order n; see Glossary). Consider

Examples 4a-e show some tetrachords from the Introduction to Igor Stravinsky's *The Rite of Spring*,[27] and Examples 5a-e provide <u, j, p>-isographic networks that interpret their pcsets. These networks do not incorporate I_n arrows. Rather, we interpret their tetrachords with T_n and M_n operators. Example 5a relates to 5b by <1, 8, p>, to 5c by <5, 0, p>, to 5d by <5, 8, p>, and to 5e by <1, 2, p>, where p is any multiple of 3 modulo 12. We cannot be more precise about the value of p at this stage, because it is used only in determining the mappings of MI_n and I_n operators, neither of which appears in the graphs of the example. In going from Example 5a to 5c, for instance, the T_5 operators of the former network map to $T_{5 \cdot 5} = T_1$ in the latter, hence u = 5. Then, the M_2 and M_{10} operators of the former map respectively to $M_{(5 \cdot 2) + 0} = M_{10}$ and $M_{(5 \cdot 10) + 0} = M_2$ in the latter, hence j = 0. Still, the choice of p does not influence either of these mappings; we could use <5,0,0>, <5,0,3>, <5,0,6> or <5,0,9> equivalently.

The networks' underlying pcsets might help to refine our specific choices for the variable p. For instance, the 4-23[0257] tetrachord of Example 5a relates to the 4-1[0123] tetrachord of Example 5c as pcsets by M_0, which is an involution. Then, if we want to draw a <u, j, p> isography between the networks in Examples 5a and c that reflects this structural property—one that is also an involution—we must choose either <5,0,0> or <5,0,6>, as <5,0,3> and <5,0,9> are both of order 4.[28]

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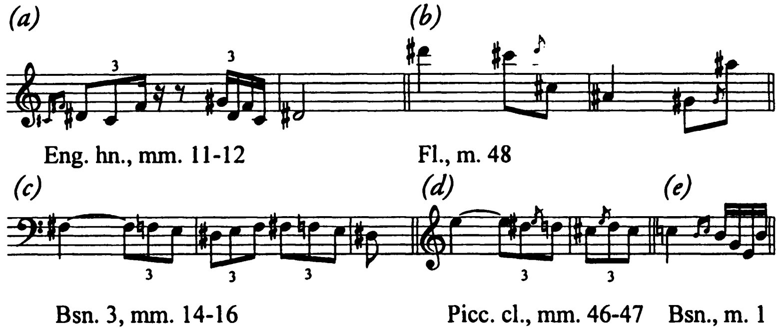
the following three subgroups: the first, generated by <1,4,0> and <5,0,0>, is isomorphic to D_6. The second, generated by <1,0,3> and <7,0,0>, is isomorphic to D_8. The third, generated by <1,6,0>, is isomorphic to C_2. Then, any member <u, j, p> of AUT(TTO_{48}) is uniquely the product of one element from each of these three subgroups.

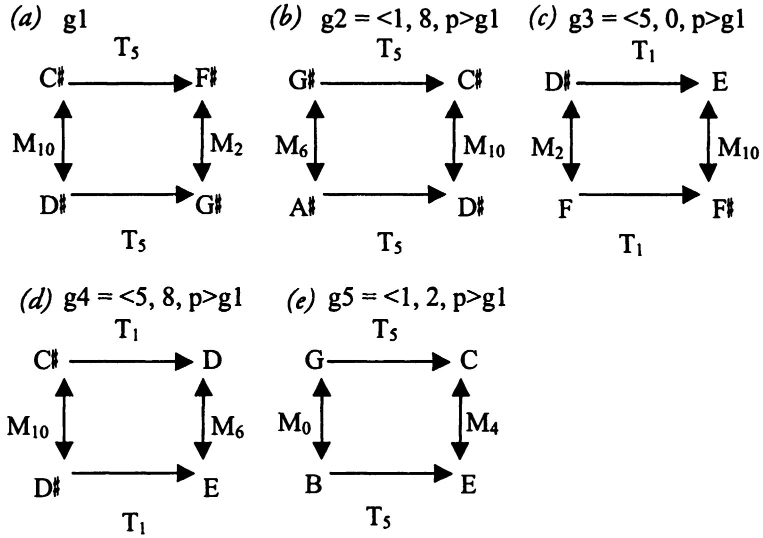
[27] Here and elsewhere in this article, we use musical examples from the Introduction to *The Rite of Spring*. We do not, however, attempt a comprehensive analysis of the Introduction, which is outside the scope of this study.

[28] The order of <u, j, p> can be determined by the following formula. Let Z_{12} be the integers modulo 12. Then, |<u, j, p>| is the least common multiple of the orders of uZ_{12}, $uZ_{12} + j$, $uZ_{12} + p$, and $uZ_{12} + j + p$. Hence, when we assert that <5,0,0> has order 2, we observe that $5Z_{12}$ has cycles of order 2 (for u = 5), and $5Z_{12} + 0 = 2$ also has cycles of order 2 (for j = p = (j + p) = 0); the least common multiple of 2 and itself is 2. Similarly, we claim |<5,0,3>| = 4. Specifically, the cycles of $5Z_{12}$ are of order 2 (for u = 5), the cycles of $5Z_{12} + 0$ are of order 2 (for j = 0), the cycles of $5Z_{12} + 3$ are of order 4 (for p = 3), and the cycles of $5Z_{12} + 0 + 3$

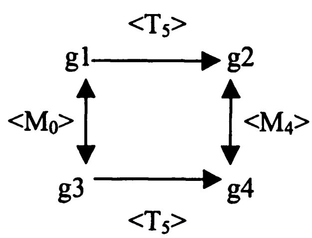
Example 4. Some tetrachords from the Introduction to The Rite of Spring.



Example 5. <u, j, p>-isographic networks of Examples 4a-e.



Example 6. Recursive hyper-network of Examples 5a-d.



Similarly, the 4-23[0257] tetrachord in Example 5a relates to the 4-23[0257] tetrachord in Example 5b by T_7 as pcsets.[29] Consequently, we might choose a <u, j, p> isography of order 12 between them to reflect the structure of T_7 as a TTO. The T_5 operators of the former network map to $T_{5 \cdot 1} = T_5$ in the latter, so u = 1. The M_2 and M_{10} operators of the former map respectively to $M_{(1 \cdot 2) + 8} = M_{10}$ and $M_{(1 \cdot 10) + 8} = M_6$ in the latter, so j = 8. Then, as <1,8,6> and <1,8,0> are of orders 6 and 3, respectively, we must decide between <1,8,9> and <1,8,3>, both of which are of order 12. We will next discover a method for refining our decisions even further.

4. The Recurse Subgroup of AUT(TTO_{48})

One of the most elegant features of traditional K-net analysis is its ability to relate networks recursively via a labeling system for isographies that shows clearly their structural identification with the twenty-four T_n and I_n operators. Klumpenhouwer gives these isographies as hyper-operators in the form <T_j> and <I_j>.[30] Under <T_j>, T_n maps onto itself, and I_n maps onto I_{n+j}; under <I_j>, T_n maps onto T_{11n}, and I_n maps onto I_{11n+j}. These twenty-four hyper-operators form a subgroup of AUT(T/I) that is isomorphic to T/I itself, showing a structural correspondence among the network level and the network-of-networks level. We will label this automorphism subgroup HYP(T/I).

HYP(T/I) is isomorphic to a particular subgroup of AUT(TTO_{48}), which we will label IMG(T/I). Using a restricted version of Lewin's <u, j, p> triple, we may give IMG(T/I)'s members as follows: u = 1 or 11; j is a multiple of 4 modulo 12;

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are of order 4 (for j + p = 3). The least common multiple of 2 and 4 is 4. We could work similar calculations for <5,0,6> and <5,0,9>.

[29] Examples 4a and b are represented in Kielian-Gilbert 1982-83, 215-16, where she finds a large-scale T_7 cycle of 4-23[0257] pcsets in the Introduction to *The Rite of Spring*. In particular, see her Examples 6c and d.

[30] Klumpenhouwer 1991. Lewin 1990, 88-89 gives these hyper-operators as <1, j> and <11, j>, respectively. Later, in Lewin 1994, he adopts Klumpenhouwer's notation.

and p is a multiple of 3 modulo 12.[31] It is important to note, however, that HYP(T/I) itself is not equal to IMG(T/I), nor is it a subgroup of AUT(TTO_{48}). The two groups HYP(T/I) and IMG(T/I) act on different sets: T/I and TTO_{48}, respectively. Nevertheless, the action of IMG(T/I) on T_n and I_n operators (disregarding its action on M_ns and MI_ns) is precisely the same as that of HYP(T/I). We will call this relation CORRELATE(T/I).

Like HYP(T/I), AUT(T/I) is isomorphic to a particular subgroup of AUT(TTO_{48}), noting again that they act on different sets. We will call this subgroup RECURSE, following Lewin 1990. We may give its members in terms of a somewhat less restricted <u, j, p> triple: u = 1, 5, 7, or 11; j is a multiple of 4 modulo 12; and p is a multiple of 3 modulo 12 (Table A.3).

To label isographies of M- and MI-inclusive networks recursively in a manner analogous to HYP(T/I), it is necessary to find a subgroup of AUT(TTO_{48}) to which TTO_{48} itself is isomorphic, and to assign hyper-operators to its members. RECURSE is precisely such a subgroup. As TTO_{48} is isomorphic to AUT(T/I), and AUT(T/I) is isomorphic to RECURSE, it follows that TTO_{48} is also isomorphic to RECURSE. (Table A.3 gives the correspondences of the members of RECURSE to <T_n>, <M_n>, <MI_n>, and <I_n> hyper-operators.)

Example 6 places the first four tetrachordal networks from Example 5 into a hyper-network. The nodes of this network represent the graphs of those tetrachords, and its arrows correspond to the unique hyper-operators from RECURSE that relate them. Moreover, the graph of Example 6 is isomorphic to any one of the graphs in Example 5, showing a degree of structural identification among the respective levels of the examples. The association of Example 5's operators with Example 6's hyper-operators allows us to refine our choice from among the possible <u, j, p> isographies we saw above—that is, from among corresponding network interpretations—more efficiently than our efforts at the end of §3. Nevertheless, Example 5e, which is isographic to the other figures, does not relate to the others in Example 5 by some member of

[31] The rationale for j's being a multiple of 4, and p's being a multiple of 3, is that any integer modulo 12 obtains uniquely by the addition of some j and p (see Lewin 1990, 119-20).

RECURSE. Consequently, we cannot as yet label its isographies to the other networks with <T_n>, <M_n>, <MI_n>, or <I_n> hyper-operators, but rather only with certain other members <u, j, p> of AUT(TTO_{48}). In the following sections, we will address this and similar issues.

5. Choosing an Appropriate Group of Transformations

Although M_n and MI_n both exchange ics 1 and 5, the order 24 T/M and T/MI subgroups of TTO_{48} have different algebraic structures. T/M is isomorphic to the abstract group $D_6 \times C_4$, while T/MI is isomorphic to $D_8 \times C_3$.[32] These distinct structures have consequences for network analysis. For example, the groups have different centers: Z(T/M) = $\{T_0,T_3,T_6,T_9\}$ is a cyclic group of order 4, while Z(T/MI) = $\{T_0,T_2,T_4,T_6,T_8,T_{10}\}$ is a cyclic group of order 6. As we will see, this variation impacts the number of distinct set-classes that may be represented by strongly isographic[33] T/M and T/MI-nets; therefore, it also impacts the K-classes to which such networks belong.[34]

Example 7 demonstrates the traditional T/I K-class for Example 2a.[35] We note that this K-class contains six different set-

[32] $D_m \times C_n$ is the direct product of the dihedral group of order m and the cyclic group of order n. That T/M is isomorphic to $D_6 \times C_4$ can be seen in the following terms: T/M contains a subgroup that may be generated by T_4 and M_0, $\{T_0, T_4, T_8, M_0, M_4, M_8\}$, and this subgroup is isomorphic to D_6. T/M also contains a subgroup that may be generated by T_3, $\{T_0, T_3, T_6, T_9\}$, isomorphic to C_4. Note that these two subgroups intersect only trivially; that is, merely in T_0, the identity element. Then, any element of T/M is the unique product of an element from one subgroup and one from the other. Similar situations relate T/MI to $D_8 \times C_3$ and TTO_{48} to $D_8 \times D_6$.

[33] Lewin 1990, 84 cites the concept of strong isography from Klumpenhouwer 1991 as follows: two networks are strongly isographic if their underlying graphs are strictly identical. "That is, the configuration of nodes and arrows are the same, and so are the transformations associated with corresponding arrows."

[34] O'Donnell 1997, 38-39 discusses K-classes. Lambert 2002, 168-71 and 191-95 revisits the issue, detailing the set-class membership of all trichordal K-classes.

[35] Lambert 2002, 191-95 implies that different interpretations of a pitch-class set may lead to different K-classes.

classes, the maximum number for a T/I K-class.[36] Lambert describes T/I K-classes in terms of cyclic wedges[37] (Example 8 shows the cyclic wedge for the networks in Example 7),[38] and "when the cyclic projections reach their midway points, they have moved from their starting points by equivalent distances, six semitones. Thus, the set-class memberships begin to repeat themselves halfway through at the tritone; any set in one half has a T_6 partner in the other. As a result, a K-class can contain representatives of no more than six different set classes."[39]

We may observe this same phenomenon in a related way. The T/I group is isomorphic to D_{24}. As such, its center, Z(T/I), is a cyclic group of order 2, and consists of $\{T_0,T_6\}$. Then, K-nets whose pcsets relate by some member in Z(T/I) (and that have the same configuration of nodes and arrows) are strongly isographic. Therefore, as the K-class is formed by a twelvefold cyclic wedge, and as Z(T/I) contains two members, we find (at most) 12/2 = 6 set-classes represented.

Similarly to Example 7, Examples 9 and 10 show K-classes for two additional interpretations of this same trichord, now using members from T/M and T/MI, respectively. As with T/I K-classes, we may observe their construction in terms of cyclic wedges. Examples 11 and 12 show these wedges for Examples 9 and 10,

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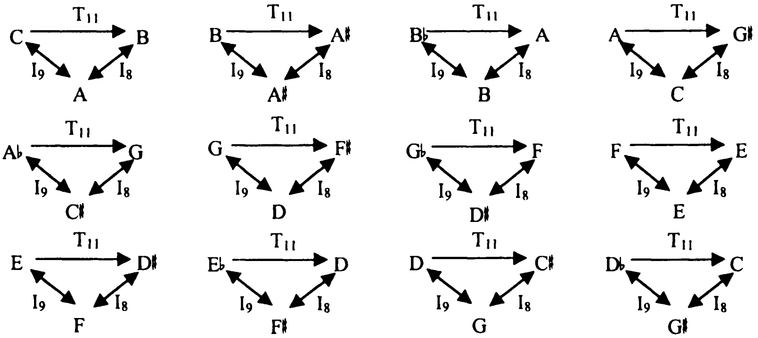
[36] Lambert 2002, 191-95 deals with K-classes primarily in terms of trichordal networks. The concept, of course, may be generalized to larger networks, as seen in the tetra- and hexachordal K-classes in his Examples 19 and 21.

[37] Lambert 2002, 169-70.

[38] Examples 7 and 8 belong to K-class 9/8 as defined in Lambert 2002, 192. Furthermore, he, along with the other authors in *Music Theory Spectrum*'s Fall 2002 issue devoted to K-nets, observes the relation of K-net isography to Perle cycles. Specifically, the latter obtain from inversionally interlocked interval cycles, such as those in the cyclic wedge of Example 8 (which belongs to Perle's [1996] row pair 9/2).

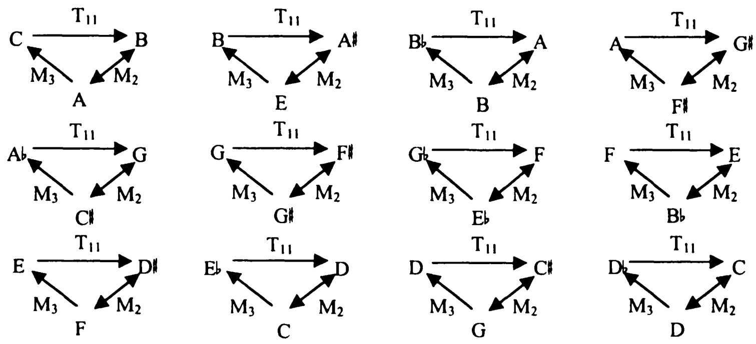
[39] Lambert 2002, 170 (n. 12) thanks an anonymous referee of his article for pointing out the "correlation between the degree of symmetry of a set-class and the multiplicity or redundancy of set-class members in each K-class."

Example 7. T/I K-class.

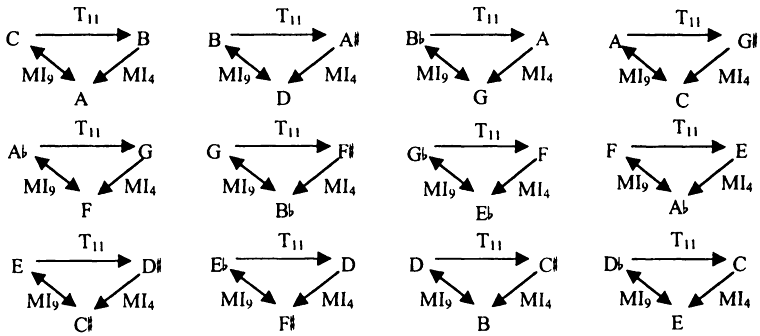


Example 8. T/I K-class as a cyclic wedge.

Example 9. T/M K-class.



Example 10. T/MI K-class.



Example 11. T/M K-class as a cyclic wedge.

Example 12. T/MI K-class as a cyclic wedge.

respectively. As in Example 8, the T_{11}-related dyad in Example 11 moves downward by semitone. However, unlike Example 8, in which the third, inversionally related pc moves upward by semitone, the remaining pc in Example 11 moves upward by perfect fifth. In Example 12, it moves upward by perfect fourth.[40] We observe that, as in the T/I K-class, the set-class memberships of the T/M and T/MI examples repeat themselves halfway through at the tritone, or every six networks. However, they also repeat more frequently: the set-classes of the T/M examples repeat every three networks, and those of the T/MIs repeat every two.

One explanation for this increase in set-class replication has to do with the centers of the various groups mentioned above. In the case of T/M, its center consists of $\{T_0,T_3,T_6,T_9\}$. Consequently, T/M networks whose pcsets are related by some member in Z(T/M) (and that have the same configuration of nodes and arrows) are strongly isographic. As a result, T/M K-classes are limited to 12/4 = 3 set-classes. In terms of T/MI K-classes, where Z(T/MI) = $\{T_0,T_2,T_4,T_6,T_8,T_{10}\}$, the set-class membership is limited even further to 12/6 = 2. Depending on our analytical purpose, we may choose to use T/M-nets or T/MI-nets to show different sets of relations. If we are seeking to show strong isographies among networks whose underlying pcsets are in a greater number of set-classes—such as in the examples we have taken from *The Rite of Spring*—then T/M-nets will be more appropriate. If our intention is instead to demonstrate a greater number of possible isographies among networks that project fewer set-classes, then T/MI networks will be more useful.[41]

6. The AUT(T/M) Group

As in their traditional T/I counterparts, isographies among T/M-nets derive from automorphisms of their underlying group of

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[40] In terms of the Perle-cyclic model, the interlocking interval cycles of Examples 11-12 are not inversionally related. They are instead related by multiplication by 5 and 7, respectively.

[41] Examples with more limited set-class membership, such as some of the pieces in Bartók's *Mikrokosmos*, might suggest the use of T/MI networks.

transformations. Hence, the structure of AUT(T/M) is of interest. Whereas AUT(T/I) contains forty-eight elements, AUT(T/M) contains only twenty-four. (Table A.4 shows its members as mappings of T/M onto itself.) Perhaps unfortunately for the analyst who is seeking recursive levels, AUT(T/M) is not isomorphic to T/M itself; rather, it is isomorphic to the abstract group $D_6 \times C_2 \times C_2$.[42] Furthermore, being the same size as T/M, it does not contain a proper subgroup that is isomorphic to T/M (as is the case with AUT(T/I), wherein the subgroup HYP(T/I) is isomorphic to T/I). As a result, we must look elsewhere for a complete source of recursive isographies—namely, a group we will call HYP(T/M).

One seemingly logical point of departure for this search might be within AUT(TTO_{48}). We recall the CORRELATE(T/I) relation from §4: T/I is isomorphic to a subgroup IMG(T/I) of AUT(TTO_{48}), and the action on T_n and I_n operators alone under IMG(T/I) is precisely the same as HYP(T/I). Similarly, we will now try to find a subgroup of AUT(TTO_{48}) that is isomorphic to T/M, and whose action on T_n and M_n operators is the same as that of AUT(T/M). We will call this relation CORRELATE(T/M). A potential place to find such a subgroup is in RECURSE, which has images of all forty-eight TTOs. One subgroup, which we will call IMG(T/M) (the first twenty-four rows in Table A.3), contains the twenty-four hyper-operators that are counterparts to the various T_ns and M_ns. Therefore, as IMG(T/M) is isomorphic to T/M, it satisfies the first part of CORRELATE(T/M).

A difficulty arises, however, when we observe IMG(T/M)'s action on T_ns and M_ns alone. First, we cannot distinguish a unique action on these operators under any member of the center of IMG(T/M), Z(IMG(T/M)) = $\{\langle T_0\rangle,\langle T_3\rangle,\langle T_6\rangle,\langle T_9\rangle\}$, or among

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[42] Consider the following three trivially intersecting subgroups of AUT(T/M): the first may be generated by the automorphisms that map T_n to T_n and M_n to M_{n+4}, and T_n to T_{5n} and M_n to M_{5n+0}. This subgroup is isomorphic to D_6. The second is generated by the automorphism that maps T_n to T_n and M_n to M_{n+6}; it is isomorphic to C_2. The third is generated by the automorphism that maps T_n to T_{7n} and M_n to M_{7n+0}; it is also isomorphic to C_2. Then, any member of AUT(T/M) may be expressed uniquely as the product of single members from each of these subgroups.

the members of any coset of Z(IMG(T/M)).[43] When we refer to Table A.3, we notice that u and j of <u, j, p>—which determine the action on T_ns and M_ns—are invariant for the members of Z(IMG(T/M)). A more difficult obstacle arises when we consider that the action of half the members of AUT(T/M) on T_ns and M_ns are not represented by the members of RECURSE.[44] RECURSE contains forty-eight members, and, as every unique action on T_ns and M_ns corresponds to four members of the group, we find only 48/4 = 12 distinct mappings of those operators; however, AUT(T/M) contains twenty-four members. Clearly, the second half of the CORRELATE(T/M) relation fails for IMG(T/M), and this failure is one reason why we could not find a hyper-operator from RECURSE that relates the tetrachordal network of Example 5e to the others in Example 5, even though they are isographic.

Of course, we might have ascertained the failure of CORRELATE(T/M) much more quickly. We could simply have pointed to the fact that AUT(T/M) contains no elements of order > 6 (Table A.4), whereas the automorphism that corresponds to T_1, for example, must be of order 12. Nonetheless, the preceding discussion illustrates certain differences between the T/M model and the more familiar and recursively convenient T/I model. Rather than mapping AUT(T/M) *onto* T/M, however, a natural method does exist of mapping a particular subgroup of AUT(T/M) *into* T/M,[45] and this process will assist us in finding appropriate labels for hyper-operators. This subgroup of AUT(T/M) is the inner automorphism group of T/M, INN(T/M) (Table A.5).

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[43] We recall that under <T_0>, <T_3>, <T_6>, or <T_9>, T_m maps to T_m and M_a maps to M_a. Similarly, if T_m maps to T_n and M_a maps to M_b under <x><T_0>, for some <x> in AUT(TTO_{48}), then T_m will also map to T_n and M_a to M_b under <x><T_3>, <x><T_6>, or <x><T_9>. The same situation obtains from multiplication on the right by <x>, by virtue of Z(IMG(T/M))'s being normal in AUT(TTO_{48}).

[44] The members of AUT(TTO_{48}) that are not in RECURSE are those <u, j, p> triples in which j equals 2, 6, or 10.

[45] "Natural" in this context is in terms of its mathematical sense: that is, mathematicians speak of the natural mapping of an algebraic structure to its quotient structures. See Dummit and Foote 1999, 84.

7. The Inner Automorphism Group INN(T/M)

Before continuing our pursuit of recursive T/M structures, we will first examine some other aspects of inner automorphisms, as they have particular analytical merits that are not present in outer automorphisms. Klumpenhouwer states that "the case for inner automorphisms is (relatively speaking) more 'phenomenologically' regulated than is the case for outer automorphisms, in the sense that it conforms more closely to a requirement that methodological structures and procedures ought to be meaningful in some extended sense to musical experience (broadly construed)."[46] The phenomenological regulation to which Klumpenhouwer refers has to do with the particular nature of isographies that derive from the inner automorphism group. Specifically, an arrow in a network between two pc nodes represents a transformation between those pcs. Now, if we wish to transform *that transformation* by another (or same) transformation, the mathematical technique for doing so is by conjugation. On a graph with arrows α and β, conjugation of α by β, $\alpha^{\beta} = \beta\alpha\beta^{-1}$, is represented by moving *the arrow* α by β.[47] Indeed, we can transform entire graphs in this way, providing for a type of isography that is less abstract than that via outer automorphisms.

We give an example from the opening bassoon solo from *The Rite of Spring* (Example 13). First, we consider the {A,B,C,D} 4-10[0235] tetrachord from m. 2. Example 14a shows one network interpretation of this collection: C descends to A via the passing tone B (we omit the embellishing grace notes), using arrows as T_n operators. The D also moves to A, either through three arrows, by way of its being an (implied) upper neighbor to C, or directly through a single arrow. We have not assigned T_n labels to the

[46] Klumpenhouwer 1998, 91-92.

[47] See Examples 1 and 2 in Klumpenhouwer 1998, 84-85. The presence of β^{-1} in this product suggests one reason why transformational schemes in music theory often incorporate algebraic group structures: conjugation, the natural method of transforming one transformation α by another β, requires the inverse of β. As group structures provide axiomatically an inverse for every element, conjugation by any member of the group is possible. This property is not a feature of simpler algebraic systems, such as groupoids, semi-groups, monoids, and sets.

arrows that connect to the node D; rather, we have used M_n arrows for reasons that will become clear later.

Now we will transform the operations in Example 14a by some member of the INN(T/M) group; for illustration, we will use conjugation by T_7, or $[T_7]$,[48] which has analytical implications that we will explore later. Under this conjugation, the T_n arrows of Example 14a remain invariant, while the M_n arrows transform to M_{n+8}. The arrows of the graph of Example 14b show the result of this transformation. Moreover, the specific node content of Example 14b is {E,F♯,G,A}, a transposition of Example 14a's pcs by T_7. As such, the respective transformations on nodes and arrows relate significantly. This correspondence is a feature of Klumpenhouwer's "phenomenological regulation": for any operation on the pcs of a network, we find a corresponding graphic transformation by conjugation. Example 14c shows this transformation on nodes and arrows for a further conjugation by T_7.

Returning to the passage, the bassoon continues in m. 4 with another C-B-A descent, now filling in the chromatic pitch B♭ (m. 4); then, it leaps down to G♭ (m. 5). In addition to the referential C-B-A motion, the three other pitches we have now named are D, B♭, and G♭. Examples 15a-c present three networks, using T_n arrows for the C-B-A descent (as in Example 14a) and M_n arrows that connect to the variable fourth pc. We note that the networks of Examples 15a-c are strongly isographic to those of Examples 14a-c, respectively. Consequently, the operators of their underlying graphs also relate to each other through a series of conjugations by T_7, demonstrating one way in which the passage's motivic growth process mirrors the T_7-cycle that others authors[49] have observed in the larger growth process of the Introduction.

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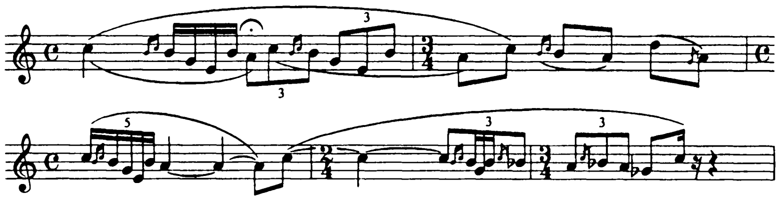
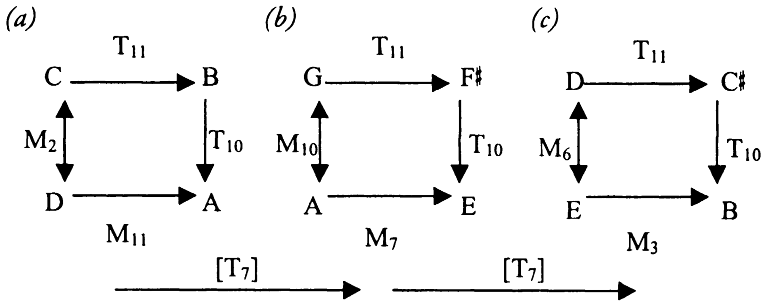
[48] Klumpenhouwer 1998 uses square brackets for hyper-operators that derive from conjugation (i.e., inner automorphisms).

[49] See Perle 1977, 10-12, Kielian-Gilbert 1982-83, 215-16, Antokoletz 1984, 313-16, and van den Toorn 1987, 178-79 and 183-88.

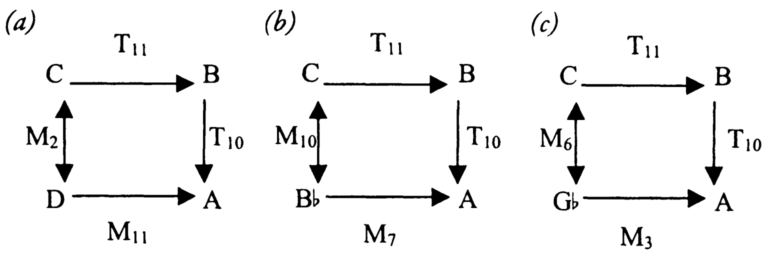
Example 13. Opening bassoon solo from The Rite of Spring.



Example 14. Three isographic networks, using [T7] from INN(T/M).



Example 15. Three isographic networks from the bassoon solo.



In Example 14a (and 15a), the specific pcs that represent the nodes of the strongly isographic networks are not the only ones that satisfy the transformational relationships given by the arrows. The K-class contains twelve networks with different node contents. We know from §5 that the K-class contains representative of three set-classes, and that for each of these set-classes, we will find four pcsets that relate to each other by T_0, T_3, T_6, and T_9. For the K-class of Example 14a, these set-classes are 4-10[0235], 4-1[0123], and T_n-class 4-13B[0356]. In fact, these three set-classes comprise the K-classes of any network conjugation of Example 14a by a T_n operator (e.g., Examples 14a-c and 15a are all members of 4-10[0235], 15b is a member of 4-1[0123], and 15c is a member of 4-13B[0356]).[50] Furthermore, we find the complete T_n-classes (for the asymmetric pcsets, set-classes for inversionally symmetric pcsets) partitioned among the three K-classes of Example 14's and Example 15's networks—four representatives of each in the three respective K-classes—allowing us to draw inner-automorphic isographies among any of their members.

George Perle comments on the opening bassoon solo and its extension in mm. 14-15 (Example 16) in terms of implied interval cycles. We find certain resonance between his remarks and our discussion. "A forms a structural minor third with C, as a part of a hexachordal collection encompassing the C to B segment of the circle of fifths. The same A then serves as a pivotal note to the minor third below. The addition of another minor third at measure 15 completes one of the three partitions, C-A-F♯-D♯, of the interval-3 cycle".[51] We note that this interval-3 cycle, as represented by T_n operators, is precisely the center of the T/M group.[52]

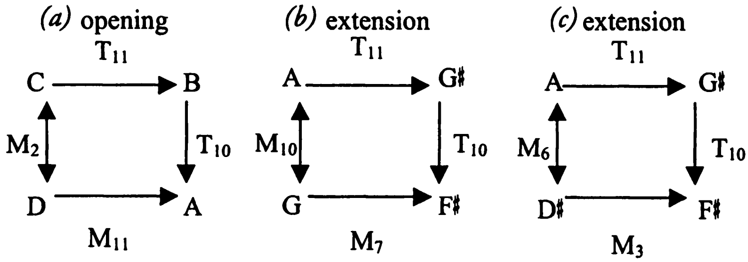
[50] The K-classes of Example 14a under conjugation by any M_n operator consist of representatives of scs 4-10[0235], 4-13A[0136], and 4-23[0257].

[51] Perle 1977, 11.

[52] Interestingly, the piccolo clarinet line in mm. 5-6, which elides with the opening bassoon solo, presents a sequential fragment of an interval-2 cycle. This cycle, as represented by T_n operators, is the center of the T/MI group.

Example 16. The extension of the bassoon solo, mm. 14-15.

Example 17. Isographic networks from the opening and the extension of the bassoon solo.



Perle continues: "Each interval is delineated in a different way, the C-A framing the diatonic scale degrees $\hat{3}$-$\hat{2}$-$\hat{1}$ (C-B-A), the A-F♯ framing a segment of the interval-1 cycle, the F♯-D♯ stating the principal cyclic interval directly and completing a second tritone, A-D♯. The figure that unfolds this symmetrically partitioned tritone is the principal motive of the 'Introduction'."[53] Specifically, the extension of the bassoon solo (mm. 14-15) is a varied transposition of mm. 4-5 down by minor third. As Perle observes, this transposition extends and completes an interval-3 cycle. We note, then, that this passage reiterates the series of T_7 conjugations we described above, as corresponding T/M-nets whose underlying pcsets transpose by some member in the center of T/M—in this case T_9—are strongly isographic. (Compare Examples 15 and 17.) Examples 17b-c parallel 15b-c down a minor third. The C-B-A referential descent of the latter is replaced by A-G♯-F♯ in the former, and the new variable fourth pitches are G-natural (m. 14) and D♯ (m. 15).

8. AUT(T/M) Revisited

As we noted at the end of §6, a natural correspondence exists between a group and its inner automophism group, and this correspondence will assist us in our pursuit of a recursive group of T/M-net transformations. Namely, we define a quotient group of T/M modulo its center, T/M / Z(T/M). This quotient group is of order 24/4 = 6, and, by a standard theorem of group theory,[54] it is isomorphic to INN(T/M). Its identity element is the center of T/M itself, the cyclic subgroup E = $\{T_0, T_3, T_6, T_9\}$. Then, we may map INN(T/M) into T/M in terms of a homomorphism F, with conjugation by the members of E as the kernel of F. Specifically, under F, the image of $[T_x]$ is T_{8x}, and the image of $[M_x]$ is M_{8x} (Table A.5).

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[53] Perle 1977, 12.

[54] e.g., Scott 1987, 50 (Theorem 3.2.4).

These automorphisms must be of the same order as their images in T/M. For example, the inner automorphism induced by T_0 (or equivalently by T_3, T_6, or T_9) is of order 1. Accordingly, this action is the identity element of INN(T/M), and corresponds to T_0, the identity element of T/M. The two other distinct automorphisms that are induced by T_x operators both have order 3, and are inverses of each other; they correlate to T_4 and T_8. Furthermore, the three distinct automorphisms induced by M_x operators all have order 2. They have images in the following M_n operators with even indices: M_0, M_4, and M_8.

Klumpenhouwer notes that the inner automorphisms $[T_x] = [T_{x+6}]$ and $[I_x] = [I_{x+6}]$ of the T/I group correlate to the members $<T_{2x}>$ and $<I_{2x}>$ of HYP(T/I), respectively.[55] Similarly, we observe that the inner automorphism $[T_x] = [T_{x+3}] = [T_{x+6}] = [T_{x+9}]$ of the T/M group corresponds to $<T_{8x}>$ of some group HYP(T/M) (which we will define fully later). The same relationship holds precisely for the indices of the $[M_{x+j}]$ and $<M_{8x}>$ hyper-operators, where j is a multiple of 3 modulo 12.

By definition, INN(T/M) of order 6 is a normal subgroup of the full automorphism group AUT(T/M) of order 24. Hence, we may define a quotient group, AUT(T/M) / INN(T/M), of order 24/6 = 4. Its members are the four cosets of INN(T/M), inclusive (Table A.6). When we adjoin the first two cosets in Table A.6, we form the largest subgroup of AUT(T/M) that is isomorphic to a subgroup of T/M. This order 12 subgroup contains the T_ns and M_ns with even indices.[56] We will label it EVEN(T/M). Examples 18a-c show three isographic T/M-nets, and Example 19 interprets these networks recursively as a hyper-network, using members of EVEN(T/M).

The two other cosets of INN(T/M) in AUT(T/M) contain together the remaining twelve automorphisms. Although they do

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55 Klumpenhouwer 1998, 87-88.

56 Two other order 12 subgroups exist in T/M: (1) the twelve T_n operators, and (2) the group that contains the T_ns with even indices and the M_ns with odd indices. Neither of these subgroups, however, is isomorphic to a subgroup of AUT(T/M).

not have images in T/M, they do in TTO_{48}.[57] When we adjoin these automorphisms to the previous twelve, we notice a one-to-one correspondence from AUT(T/M) to the order 24 subgroup of TTO_{48} with even indices, which we will label EVEN(TTO_{48}). As a result, we have now been able to assign hyper-operators (albeit some with images outside the T/M group) to all members of AUT(T/M), but we are still able to model only whole-tone segments recursively.

9. SPIN and Order 12 Transformations on Trichordal T/M Graphs

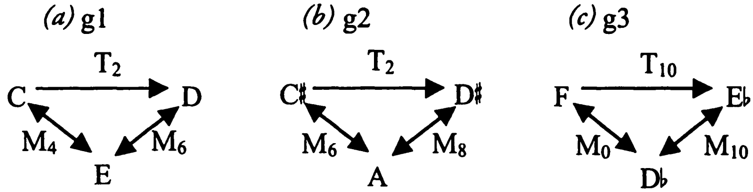
In this section, we will confine ourselves to trichordal T/M-nets, partly because they are the easiest to visualize, but also because any larger network can be deconstructed into trichordal subnetworks. We will consider larger networks in the following section.

The problem of limiting ourselves to the $<T_j>$ and $<M_j>$ operators above is that we cannot construct hyper-networks that contain hyper-operators with odd indices. For example, we are not even able to model recursively a motive as basic as a [013] trichord. The reason for this failure is simply that no order 12 or 4 automorphisms of T/M exist to correlate to the TTOs of order 12 (T_1, T_5, T_7, and T_{11}) and order 4 (T_3 and T_9, and all M_ns where n is odd). Therefore, we must look elsewhere for a fully recursive system—one that does not have automorphisms as its basis, and that consequently does not describe isography in the sense of Lewin. We define here a recursive system that uses the mappings of EVEN(T/M) as its point of departure, but describes the remaining transformations merely on graphs, and not in terms of automorphisms of the T/M group.

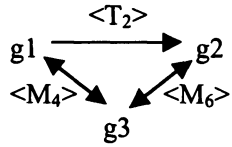
Such graphic transformations are reminiscent of Stoeker's "axial isographies" among T/I K-nets, which do not derive from

[57] This situation recalls AUT(T/I)'s containing twenty-four members outside HYP(T/I) with images in TTO_{48} as M_ns and MI_ns.

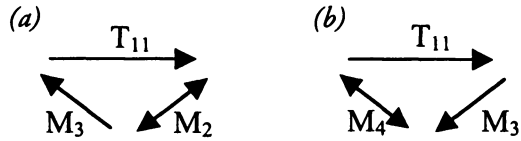
Example 18. Isographic T/M-nets.



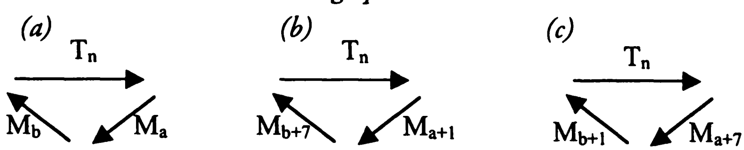
Example 19. Recursive hyper-network of Examples 18a-c.



Example 20. (a) Well-formed and (b) ill-formed T/M graphs.



Example 21. Order 12 graphic transformations on trichordal T/M graphs.



automorphisms of the T/I group.[58] In trichordal axial isography, one I_n arrow remains invariant, while the remaining I_n and T_n arrows vary. These transformations are generally not true graph isomorphisms, particularly when a T_n operator maps to another of a different order (e.g., when T_3 maps to T_4). In making the transition from the group-theoretical to the graph-theoretical model, we lose the following properties of groups as axioms: closure under the group operation, associativity, presence of an identity, and presence of an inverse for each operation (arrow). We may nevertheless define graphic systems that have some or all of these properties. For instance, Stoecker's axial isographies are in fact closed under composition, they are associative, and they contain an identity and inverse elements. We also gain some freedom: we are no longer restricted by the limitations of group structures and their symmetries. Indeed, in pursuing recursion, these structures and symmetries are not necessarily the only bases on which higher-level interpretive acts may take place.

To describe a trichordal network transformation of order 12 to correlate to $<T_1>$, we need to divide $<T_2>$ by two; but when we substitute 1 (or any other odd number) for j in $<T_j>$, and apply this hyper-operator to a well-formed graph, the resulting graph is not well-formed. For instance, Example 20a is well-formed, but Example 20b, which performs +1 on both the indices of Example 20a's M_n operators, is not. Whereas $M_3M_2 = T_1$, $M_4M_3 \neq T_1$. Nonetheless, we may express any even integer j modulo 12 as 2k in precisely two ways: k = j/2, and k = (j/2) + 6. In other words, as 2 is the sum of either 1 + 1 or 7 + 7 modulo 12, we may define a transformation *on the graphs* of trichordal networks that sends the index of the operator of one M_n arrow by +1, and the other by +7 (while keeping the index of the T_n arrow invariant). Such a transformation is of order 12, and performed twice obtains $<T_2>$. Moreover, it also yields a well-formed graph. It is important to note, however, that this transformation does not derive from an automorphism of T/M, particularly as M_n is always of a different order from M_{n+1} and M_{n+7}.

[58] Stoecker 2002.

In this manner, any trichordal T/M-net with arrows M_a and M_b (Example 21a) may transform in two different ways: either M_a goes to M_{a+1} and M_b goes to M_{b+7}, or M_a goes to M_{a+7} and M_b goes to M_{b+1} (Examples 21b-c). To be consistent in our analyses, first we need to be consistent in the way in which we conceive of the graphs—meaning here the directions of their arrows—and then we need to define which of these two transformations will represent $<T_1>$. (The other will represent $<T_7>$). Changing the direction of a graph's arrows, while preserving their respective associations as T_n or M_n types, does not affect the overall GIS interval content of any corresponding network (see n. 13). Therefore, we may *write* our graphs with the arrows pointing in whatever directions we require to project our interpretation of the pcs, networks, and so on, that we associate with its nodes. Nevertheless, when we perform a $<T_1>$ transformation on a trichordal T/M-net, we will *conceive* of its graph as being ordered clockwise—that is, all arrows point clockwise.[59] We will call this conception the SPIN of the graph. Then, for consistency, we will always increment the index of the operator of the M_n arrow that follows the T_n arrow (clockwise) by +1, and that of the other by +7 (Example 21b). Later, we can readjust the arrow directions to suit our interpretation.

This $<T_1>$ performed twice yields $<T_2>$ as we defined above, since $M_{a+1+1} = M_{a+2}$, and $M_{b+7+7} = M_{b+2}$. $<T_1>$ reiterated three times sends the index of the first M_n arrow to n + 1 + 1 + 1 (or n + 3), and that of the other to n + 7 + 7 + 7 (or n + 9), resulting in $<T_3>$. $<T_1>$ performed four times equals $<T_4>$ as defined above, and so forth. Furthermore, $<T_1>$ and its powers may be multiplied to $<M_0>$ above, thereby generating a group of twenty-four trichordal graphic transformations that is indeed isomorphic to T/M itself.[60] We will call this group HYP(T/M). Thus, we have finally defined a fully recursive system for relating trichordal T/M-nets.

Now we may model a [013] trichord recursively. Example 22a returns to the C-B-A referential descent from the *The Rite of Spring*. We take the melodically salient C-B to be the interval we represent

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[59] For now, we will even conceive of involutions as directed intervals, moving clockwise.

[60] We recall that the T/M group is generable from T_1 and M_0.

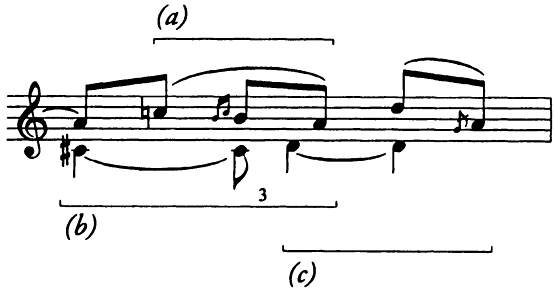
with a T_n arrow, T_{11}. Then, we give B-A and C-A as M_2 and M_9, respectively (Example 23a). The first harmonic interval of the piece occurs in m. 2. The horn C♯ anticipates, then sounds against, the C-natural of the bassoon solo, thereby creating a dissonant major seventh and a major-minor clash with the A to which the C descends (Example 22b). We represent these intervals as follows: C♯-C as T_{11}, and C-A and C♯-A as M_9 and M_4, respectively (Example 23b). Finally, the horn D anticipates and supports an A-D-G melodic figure (Example 22c). We represent the D-A interval (which occurs both melodically and harmonically) as T_7, and the A-G and D-G intervals as M_{10} and M_9, respectively (Example 23c).

All three networks relate to each other in significant ways. We interpret each to contain one interval, either T_{11} or T_7, which generates a cycle of order 12, and two additional intervals of orders 2 and 4 (M_a and M_b, where a is even and b is odd). Within each network, each interval is equivalent to the product of the other two—hence they are well-formed—and these products correspond among the three. Even though these relationships are not isographies in a traditional K-net theoretical sense, they have many of the same properties, and we can view them as hyper-operators.

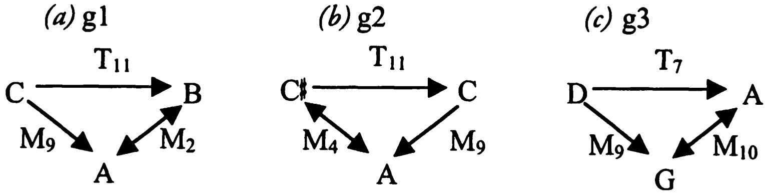
For example, our interpretation suggests that the T_{11}-related dyad C-B in Example 22a ascends (in pc space) to C♯-C in Example 22b, while the M_n-related A remains fixed. This action generates an order 12 cycle, and conforms to our notion of $<T_7>$ in HYP(T/M).[61] Similarly, the T_{11}-related C-B semitone expands to a T_7-related D-A fourth at its climax (Example 22c), while the earlier B-A (M_2) approach to A from above rotates to G-A (M_{10}) from below. This transformation is an involution, and it conforms to $<M_0>$. Furthermore, Example 23b relates to Example 23c by $<M_1>$, which is a product of these previous two hyper-operators, $<M_0><T_7>^{-1} = <M_1>$. Accordingly, we may construct a hyper-network (Example 24), which demonstrates the same structural properties as each of the pc networks. Example 24 relates these trichords recursively via HYP(T/M), even where we cannot

[61] To see the $<T_7>$, we must SPIN the trichordal networks clockwise. We have reoriented the arrows in the graphs in Example 24 to reflect our interpretation of the trichords.

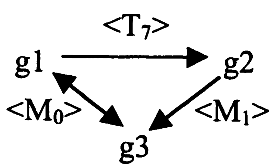
Example 22. Three trichords in m. 2.



Example 23. Isographic trichordal networks from Example 22.



Example 24. Recursive hyper-network of Examples 23a-c.



do so in terms of AUT(T/M). In particular, we note the hyper-operators with odd indices.

10. HYP(T/M) Applied to Tetrachordal and Larger Networks

In the previous section, we alluded to the fact that larger networks may be deconstructed into trichordal subnetworks. The implication was that we might apply the operators of HYP(T/M) to these subnetworks, and thereby transform the larger networks. Even though this assumption is mostly true, certain difficulties arise; consequently, a full generalization of this process is outside the scope of this study. However, we will suggest one way in which we might apply this method to tetrachordal networks, and then address briefly still larger networks.

The graph of any tetrachordal network contains six arrows (although some may be merely implied).[62] We may model it accordingly as a tetrahedron. We find two classes of well-formed T/M tetrachordal graphs. First are those graphs that contain three T_n operators and three M_n operators. In any such tetrahedral graph, one face represents an L-net, and the other three faces represent trichordal T/M-nets. We will call this set of graphs Class 1. Second, we find examples that contain two T_n operators and four M_n operators. In these tetrahedral graphs, all four faces represent trichordal T/M-nets. These graphs will comprise Class 2.

In applying some member, either $<T_n>$ or $<M_n>$, of HYP(T/M) to any such network, it would be most efficient to apply it consistently to all faces of the tetrahedron, if possible. Given an appropriate SPIN, this scheme can be made to work for Class 1 graphs, thereby obtaining well-formed tetrachordal networks. However, it does not work for Class 2. Here we must apply $<T_n>$ to two faces, but then $<T_{7n}>$ to the other two (similarly, we must apply $<M_n>$ and $<M_{7n}>$ to two faces each). While this situation is not problematic for hyper-operators with even indices, which map onto themselves under modulo 12 multiplication by 7,

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[62] A network with n nodes will always have precisely 1 + 2 + ... + (n - 1) arrows, just as the interval-class vector for a pcset of cardinality n accounts for 1 + 2 + ... + (n - 1) intervals.

it is a hindrance for hyper-operators with odd indices. Namely, how do we decide to which faces we apply the desired hyper-operator? We may overcome this obstacle by *conceiving* our graphs in a particular way, then performing the hyper-operation.

For this purpose, we will depict our tetrachordal T/M-nets as quadrangles, rather than as tetrahedra. When so configured, network analysts show typically the four outer arrows, and not the two internal composite arrows, as they are implicit from the outer arrows. In Class 1 graphs, we will position two of the T_n arrows along the top and left edges; in Class 2 graphs, we will give all outer edges as M_n arrows (Example 25). Again, for consistency, we will define a particular SPIN for our graphs. This orientation of arrow directions may or may not conform to our interpretation of a network's underlying pcset, but it will allow us to perform a transformation on its graph. We may reorient the arrows at a later stage. We will approach the two classes of tetrachordal graphs in turn.

First, we consider those tetrachordal networks represented by the graph of Example 25a (Class 1). We require a consistent set of arrow directions to determine the graph's SPIN.[63] Therefore, we will always orient the T_n arrow across the top of the graph left to right, the M_n arrow on the right edge downward, and the (hidden) arrow of the composition of these two arrows upward to the left. (See the right subgraph in Example 26.) In other words, the SPIN of this trichordal subgraph is clockwise. The SPIN of the other, adjacent trichordal subgraph is, then, counter-clockwise, and its arrows point in that direction (the left subgraph in Example 26).[64] Now we transform each trichordal subnetwork according to the procedure for $<T_n>$ or $<M_n>$ above (Example 27 shows this procedure for $<T_1>$). Then we reassemble the tetrachordal graph (Example 28), and we may reorient the arrow directions to fit our interpretation of the music.

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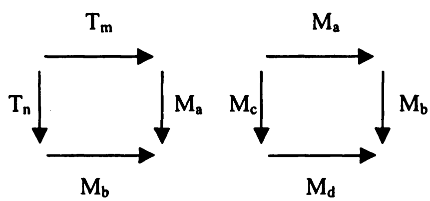
63 Again, when the order of an arrow's operator is 2—and hence is a double arrow—we will conceive of it in terms of the following directions to determine SPIN.

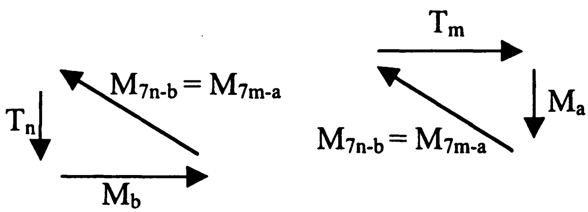
64 Such a scheme maximizes the energy of the SPIN according to the one-dimensional Ising model of physics. Douthett and Krantz 1996 discuss the Ising model in terms of maximally even collections.

Example 25. The two classes of tetrachordal T/M graphs.

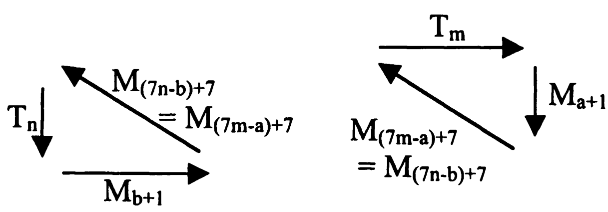
(a) Class 1 *(b)* Class 2



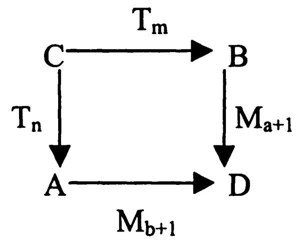
Example 26. Two trichordal subgraphs within a tetrachordal graph (Class 1).



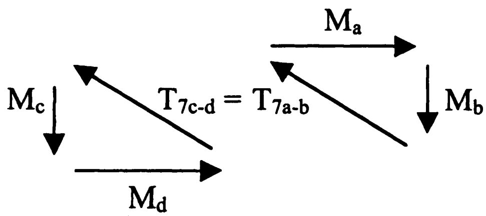
Example 27. Trichordal subgraphs under <T1> (Class 1).



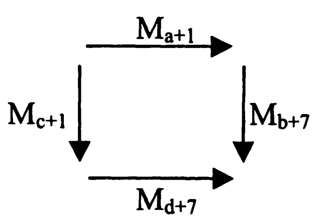
Example 28. Reassembled tetrachordal graph (Class 1).



Example 29. Two trichordal subgraphs within a tetrachordal graph (Class 2).



Example 30. Reassembled graph (Class 2).



Second, we consider networks in Class 2 (Example 25b). The two T_n arrows in their graphs are the composite internal arrows. We will divide these networks into trichordal subgraphs, orient them, and determine their SPIN, as follows. The top M_n arrow will always point left to right. The rightmost M_n arrow will always point downward, and their composite T_n arrow will always point diagonally upward to the left. Therefore, the SPIN of this trichordal subgraph is clockwise. As with the previous class of tetrachordal networks, the SPIN of the other trichordal subgraph is counterclockwise (Example 29). To effect a $\langle T_1 \rangle$ transformation to the overall graph, we perform $\langle T_1 \rangle$ on each respective trichordal subgraph, then reassemble (Example 30) and reorient the arrow directions to demonstrate best our interpretation.

The procedures we outline here may extend to pentachordal and other larger networks, as any such network can be deconstructed into trichordal subgraphs. Moreover, each of these subgraphs, given an appropriate SPIN, can transform by some member of HYP(T/M) and reassemble to create a new network that is well-formed. In such a procedure, it is important to define carefully the specific orientations of the graph's arrows that we will use in the SPIN, and to use those orientations consistently, especially in implementing hyper-operators with odd indices. Then, reorientation of the networks may reflect our interpretation of their underlying pcsets. Nevertheless, we recall that the system we propose here for tetrachordal and larger networks is not a full generalization, and leaves several questions unaddressed.

11. Conclusions

The foregoing musical and mathematical analysis has proposed a fully recursive system for using T- and M-inclusive K-nets. Because no (sub)group of automorphisms of the T/M group is isomorphic to T/M itself, we have had to define a group of transformations on graphs that is isomorphic to T/M. This group allowed us to model pitch-class networks recursively as networks-of-networks, and so on. In doing so, we gave initially a graphic transformation on trichordal networks. Then we observed how we may deconstruct larger networks into trichordal subnetworks, which, under these graphic transformations, may be reassembled to produce a transformation on the larger network. However, we left the full generalization of this process for future work.

We observe now some potential modifications and extensions to the theory. The particular system we used to deconstruct larger networks into trichordal ones is not the only method for doing so, nor are the specific orientations we gave as the SPIN of these graphs the only ones available. Nonetheless, the deconstructions and SPINs that we have defined do, in fact, result in well-formed networks under the members of the group of graphic transformations. Though we may describe other related systems, they would also have to result in well-formed networks.

We have concentrated primarily on the T/M group. As we suggested earlier, similar systems can be described for the T/MI group, as well as for other groups of operations.[65] The T/M and T/MI groups (alone, but not together) are particularly salient in the analysis of music that involves exchanges of chromatic and circle-of-fifths spaces, but that does not incorporate inversion structurally. We have pointed out that the primary distinction between the use of T/M and T/MI-nets is that the former system contains a larger variety of set-classes in its various K-classes, while the latter describes a greater number of strong isographies among fewer set-classes.

We have drawn our musical examples from the Introduction to Stravinsky's *The Rite of Spring*, as it provides an excellent illustration of a piece of music that fits the above criteria for network analysis with the T/M group. One might even undertake a more substantial analysis of the Introduction, or of the entire ballet, using this method. In fact, a large body of twentieth-century music lends itself well to analysis with T/M-nets, such as certain works by Debussy, Bartók, Lutosławski, and Crumb, among others, in which diatonic and chromatic materials coexist.

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[65] For example, Gollin 1998 describes a recursive system of isography with the Q/X group. We could explore various Q/M hybrids, some of which have quaternion and quaternion-like algebraic structures, or other groups that involve M and MI operators, as well.

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Glossary

automorphism. An isomorphism of a group to itself.

automorphism group. The set of all automorphisms of a group. Automorphism groups themselves have group structures.

center. The normal subgroup of elements within a group that commute with all elements in the group.

conjugation of a by b (elementwise). The composition bab^{-1}.

conjugation of G by b (groupwise). The set of elementwise conjugations $\{bgb^{-1} \mid$ for all g in G$\}$.

coset. The set of products of a subgroup's elements and a group element, multiplied consistently on the left or the right.

cyclic group of order n (C_n). The group of rotational symmetries for a regular n-gon.

dihedral group of order n (D_n). The group of symmetries (rotations and reflections) for a regular n/2-gon. (N.B.: Some sources, including Dummit and Foote (1999), give the dihedral group of order n as $D_{n/2}$.)

direct product. The set of all products among respective members of a family of sets. The direct product of two or more groups is itself a group.

homomorphism. A function F that maps a group G to another H, such that $F(g)F(h) = F(gh)$, for all g and h in G.

inner automorphism. An automorphism that obtains via groupwise conjugation.

inner automorphism group. The set of all inner automorphisms of a group. Inner automorphism groups themselves have group structures.

involution. A group element of order 2.

isomorphism. A homomorphism that is one-to-one and onto.

kernel. In an homomorphism $F : G \rightarrow H$, the set of elements in G that map to the identity element of H.

order. The least power of a group element that obtains the identity element.

outer automorphism. Any automorphism that does not obtain via group conjugation.

normal subgroup. A subgroup N of G, such that $gN = Ng$, for all g in G.

quotient group. A quotient (or factor) group G/N is a group structure on the cosets of a normal subgroup N in G. The group operation is $(Nx)(Ny) = N(xy)$, for any x and y in G. (As N is normal, it does not matter whether we consider left or right cosets.)

APPENDIX: TABLES

A.1. The forty-eight TTOs in cyclic notation.

Label	Cycle(s) on pcs
T_0	(0)(1)(2)(3)(4)(5)(6)(7)(8)(9)(10)(11)
T_1	(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11)
T_2	(0, 2, 4, 6, 8, 10)(1, 3, 5, 7, 9, 11)
T_3	(0, 3, 6, 9)(1, 4, 7, 10)(2, 5, 8, 11)
T_4	(0, 4, 8)(1, 5, 9)(2, 6, 10)(3, 7, 11)
T_5	(0, 5, 10, 3, 8, 1, 6, 11, 4, 9, 2, 7)
T_6	(0, 6)(1, 7)(2, 8)(3, 9)(4, 10)(5, 11)
T_7	(0, 7, 2, 9, 4, 11, 6, 1, 8, 3, 10, 5)
T_8	(0, 8, 4) (1, 9, 5)(2, 10, 6)(3, 11, 7)
T_9	(0, 9, 6, 3)(1, 10, 7, 4)(2, 11, 8, 5)
T_{10}	(0, 10, 8, 6, 4, 2)(1, 11, 9, 7, 5, 3)
T_{11}	(0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1)
M_0	(0)(1, 5)(2, 10)(3)(4, 8)(6)(7, 11)
M_1	(0, 1, 6, 7)(2, 11, 8, 5)(3, 4, 9, 10)
M_2	(0, 2)(1, 7)(3, 5)(4, 10)(6, 8)(9, 11)
M_3	(0, 3, 6, 9)(1, 8, 7, 2)(4, 11, 10, 5)
M_4	(0, 4)(1, 9)(2)(3, 7)(5)(6, 10)(8)(11)
M_5	(0, 5, 6, 11)(1, 10, 7, 4)(2, 3, 8, 9)
M_6	(0, 6)(1, 11)(2, 4)(3, 9)(5, 7)(8, 10)
M_7	(0, 7, 6, 1)(2, 5, 8, 11)(3, 10, 9, 4)
M_8	(0, 8)(1)(2, 6)(3, 11)(4)(5, 9)(7)(10)
M_9	(0, 9, 6, 3)(1, 2, 7, 8)(4, 5, 10, 11)
M_{10}	(0, 10)(1, 3)(2, 8)(4, 6)(5, 11)(7, 9)
M_{11}	(0, 11, 6, 5)(1, 4, 7, 10)(2, 9, 8, 3)
MI_0	(0)(1, 7)(2)(3, 9)(4)(5, 11)(6)(8)(10)
MI_1	(0, 1, 8, 9, 4, 5)(2, 3, 10, 11, 6, 7)
MI_2	(0, 2, 4, 6, 8, 10)(1, 9, 5)(3, 11, 7)
MI_3	(0, 3)(1, 10)(2, 5)(4, 7)(6, 9)(8, 11)
MI_4	(0, 4, 8)(1, 11, 9, 7, 5, 3)(2, 6, 10)
MI_5	(0, 5, 4, 9, 8, 1)(2, 7, 6, 11, 10, 3)
MI_6	(0, 6)(1)(2, 8)(3)(4,10)(5)(7)(9)(11)
MI_7	(0, 7, 8, 3, 4, 11)(1, 2, 9, 10, 5, 6)

MI_8	(0, 8, 4)(1, 3, 5, 7, 9, 11)(2, 10, 6)
MI_9	(0, 9)(1, 4)(2, 11)(3, 6)(5, 8)(7, 10)
MI_{10}	(0, 10, 8, 6, 4, 2)(1, 5, 9)(3, 7, 11)
MI_{11}	(0, 11, 4, 3, 8, 7)(1, 6, 5, 10, 9, 2)
I_0	(0)(1, 11)(2, 10)(3, 9)(4, 8)(5, 7)(6)
I_1	(0, 1)(2, 11)(3, 10)(4, 9)(5, 8)(6, 7)
I_2	(0, 2)(1)(3, 11)(4, 10)(5, 9)(6, 8)(7)
I_3	(0, 3)(1, 2)(4, 11)(5, 10)(6, 9)(7, 8)
I_4	(0, 4)(1, 3)(2)(5, 11)(6, 10)(7, 9)(8)
I_5	(0, 5)(1, 4)(2, 3)(6, 11)(7, 10)(8, 9)
I_6	(0, 6)(1, 5)(2, 4)(3)(7, 11)(8, 10)(9)
I_7	(0, 7)(1, 6)(2, 5)(3, 4)(8, 11)(9, 10)
I_8	(0, 8)(1, 7)(2, 6)(3, 5)(4)(9,11)(10)
I_9	(0, 9)(1, 8)(2, 7)(3, 6)(4, 5)(10, 11)
I_{10}	(0, 10)(1, 9)(2, 8)(3, 7)(4, 6)(5)(11)
I_{11}	(0, 11)(1, 10)(2, 9)(3, 8)(4, 7)(5, 6)

A.2. The rules of combination for T_n, M_n, MI_n, and I_n operators.

$T_mT_n = T_{m+n}$	$M_mT_n = M_{m+5n}$	$MI_mT_n = MI_{m+7n}$	$I_mT_n = I_{m+11n}$
$T_mM_n = M_{m+n}$	$M_mM_n = T_{m+5n}$	$MI_mM_n = I_{m+7n}$	$I_mM_n = MI_{m+11n}$
$T_mMI_n = MI_{m+n}$	$M_mMI_n = I_{m+5n}$	$MI_mMI_n = T_{m+7n}$	$I_mMI_n = M_{m+11n}$
$T_mI_n = I_{m+n}$	$M_mI_n = MI_{m+5n}$	$MI_mI_n = M_{m+7n}$	$I_mI_n = T_{m+11n}$

A.3. The members of RECURSE.

<u, j, p>	Action on TTO_{48}	Hyper-operator
<1,0,0>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+0}$, $MI_n \rightarrow MI_{n+0}$, $I_n \rightarrow I_{n+0+0}$	$<T_0>$
<1,4,9>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+4}$, $MI_n \rightarrow MI_{n+9}$, $I_n \rightarrow I_{n+4+9}$	$<T_1>$
<1,8,6>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+8}$, $MI_n \rightarrow MI_{n+6}$, $I_n \rightarrow I_{n+8+6}$	$<T_2>$
<1,0,3>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+0}$, $MI_n \rightarrow MI_{n+3}$, $I_n \rightarrow I_{n+0+3}$	$<T_3>$
<1,4,0>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+4}$, $MI_n \rightarrow MI_{n+0}$, $I_n \rightarrow I_{n+4+0}$	$<T_4>$
<1,8,9>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+8}$, $MI_n \rightarrow MI_{n+9}$, $I_n \rightarrow I_{n+8+9}$	$<T_5>$
<1,0,6>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+0}$, $MI_n \rightarrow MI_{n+6}$, $I_n \rightarrow I_{n+0+6}$	$<T_6>$
<1,4,3>	$T_n \rightarrow T_n$, $M_n \rightarrow M_{n+4}$, $MI_n \rightarrow MI_{n+3}$, $I_n \rightarrow I_{n+4+3}$	$<T_7>$

<1,8,0>	$T_n \to T_n, M_n \to M_{n+8}, MI_n \to MI_{n+0}, I_n \to I_{n+8+0}$	$\langle T_8 \rangle$
<1,0,9>	$T_n \to T_n, M_n \to M_{n+0}, MI_n \to MI_{n+9}, I_n \to I_{n+0+9}$	$\langle T_9 \rangle$
<1,4,6>	$T_n \to T_n, M_n \to M_{n+4}, MI_n \to MI_{n+6}, I_n \to I_{n+4+6}$	$\langle T_{10} \rangle$
<1,8,3>	$T_n \to T_n, M_n \to M_{n+8}, MI_n \to MI_{n+3}, I_n \to I_{n+8+3}$	$\langle T_{11} \rangle$
<5,0,0>	$T_n \to T_{5n}, M_n \to M_{5n+0}, MI_n \to MI_{5n+0}, I_n \to I_{5n+0+0}$	$\langle M_0 \rangle$
<5,4,9>	$T_n \to T_{5n}, M_n \to M_{5n+4}, MI_n \to MI_{5n+9}, I_n \to I_{5n+4+9}$	$\langle M_1 \rangle$
<5,8,6>	$T_n \to T_{5n}, M_n \to M_{5n+8}, MI_n \to MI_{5n+6}, I_n \to I_{5n+8+6}$	$\langle M_2 \rangle$
<5,0,3>	$T_n \to T_{5n}, M_n \to M_{5n+0}, MI_n \to MI_{5n+3}, I_n \to I_{5n+0+3}$	$\langle M_3 \rangle$
<5,4,0>	$T_n \to T_{5n}, M_n \to M_{5n+4}, MI_n \to MI_{5n+0}, I_n \to I_{5n+4+0}$	$\langle M_4 \rangle$
<5,8,9>	$T_n \to T_{5n}, M_n \to M_{5n+8}, MI_n \to MI_{5n+9}, I_n \to I_{5n+8+9}$	$\langle M_5 \rangle$
<5,0,6>	$T_n \to T_{5n}, M_n \to M_{5n+0}, MI_n \to MI_{5n+6}, I_n \to I_{5n+0+6}$	$\langle M_6 \rangle$
<5,4,3>	$T_n \to T_{5n}, M_n \to M_{5n+4}, MI_n \to MI_{5n+3}, I_n \to I_{5n+4+3}$	$\langle M_7 \rangle$
<5,8,0>	$T_n \to T_{5n}, M_n \to M_{5n+8}, MI_n \to MI_{5n+0}, I_n \to I_{5n+8+0}$	$\langle M_8 \rangle$
<5,0,9>	$T_n \to T_{5n}, M_n \to M_{5n+0}, MI_n \to MI_{5n+9}, I_n \to I_{5n+0+9}$	$\langle M_9 \rangle$
<5,4,6>	$T_n \to T_{5n}, M_n \to M_{5n+4}, MI_n \to MI_{5n+6}, I_n \to I_{5n+4+6}$	$\langle M_{10} \rangle$
<5,8,3>	$T_n \to T_{5n}, M_n \to M_{5n+8}, MI_n \to MI_{5n+3}, I_n \to I_{5n+8+3}$	$\langle M_{11} \rangle$
<7,0,0>	$T_n \to T_{7n}, M_n \to M_{7n+0}, MI_n \to MI_{7n+0}, I_n \to I_{7n+0+0}$	$\langle MI_0 \rangle$
<7,4,9>	$T_n \to T_{7n}, M_n \to M_{7n+4}, MI_n \to MI_{7n+9}, I_n \to I_{7n+4+9}$	$\langle MI_1 \rangle$
<7,8,6>	$T_n \to T_{7n}, M_n \to M_{7n+8}, MI_n \to MI_{7n+6}, I_n \to I_{7n+8+6}$	$\langle MI_2 \rangle$
<7,0,3>	$T_n \to T_{7n}, M_n \to M_{7n+0}, MI_n \to MI_{7n+3}, I_n \to I_{7n+0+3}$	$\langle MI_3 \rangle$
<7,4,0>	$T_n \to T_{7n}, M_n \to M_{7n+4}, MI_n \to MI_{7n+0}, I_n \to I_{7n+4+0}$	$\langle MI_4 \rangle$
<7,8,9>	$T_n \to T_{7n}, M_n \to M_{7n+8}, MI_n \to MI_{7n+9}, I_n \to I_{7n+8+9}$	$\langle MI_5 \rangle$
<7,0,6>	$T_n \to T_{7n}, M_n \to M_{7n+0}, MI_n \to MI_{7n+6}, I_n \to I_{7n+0+6}$	$\langle MI_6 \rangle$
<7,4,3>	$T_n \to T_{7n}, M_n \to M_{7n+4}, MI_n \to MI_{7n+3}, I_n \to I_{7n+4+3}$	$\langle MI_7 \rangle$
<7,8,0>	$T_n \to T_{7n}, M_n \to M_{7n+8}, MI_n \to MI_{7n+0}, I_n \to I_{7n+8+0}$	$\langle MI_8 \rangle$
<7,0,9>	$T_n \to T_{7n}, M_n \to M_{7n+0}, MI_n \to MI_{7n+9}, I_n \to I_{7n+0+9}$	$\langle MI_9 \rangle$
<7,4,6>	$T_n \to T_{7n}, M_n \to M_{7n+4}, MI_n \to MI_{7n+6}, I_n \to I_{7n+4+6}$	$\langle MI_{10} \rangle$
<7,8,3>	$T_n \to T_{7n}, M_n \to M_{7n+8}, MI_n \to MI_{7n+3}, I_n \to I_{7n+8+3}$	$\langle MI_{11} \rangle$
<11,0,0>	$T_n \to T_{11n}, M_n \to M_{11n+0}, MI_n \to MI_{11n+0}, I_n \to I_{11n+0+0}$	$\langle I_0 \rangle$
<11,4,9>	$T_n \to T_{11n}, M_n \to M_{11n+4}, MI_n \to MI_{11n+9}, I_n \to I_{11n+4+9}$	$\langle I_1 \rangle$
<11,8,6>	$T_n \to T_{11n}, M_n \to M_{11n+8}, MI_n \to MI_{11n+6}, I_n \to I_{11n+8+6}$	$\langle I_2 \rangle$
<11,0,3>	$T_n \to T_{11n}, M_n \to M_{11n+0}, MI_n \to MI_{11n+3}, I_n \to I_{11n+0+3}$	$\langle I_3 \rangle$
<11,4,0>	$T_n \to T_{11n}, M_n \to M_{11n+4}, MI_n \to MI_{11n+0}, I_n \to I_{11n+4+0}$	$\langle I_4 \rangle$
<11,8,9>	$T_n \to T_{11n}, M_n \to M_{11n+8}, MI_n \to MI_{11n+9}, I_n \to I_{11n+8+9}$	$\langle I_5 \rangle$
<11,0,6>	$T_n \to T_{11n}, M_n \to M_{11n+0}, MI_n \to MI_{11n+6}, I_n \to I_{11n+0+6}$	$\langle I_6 \rangle$
<11,4,3>	$T_n \to T_{11n}, M_n \to M_{11n+4}, MI_n \to MI_{11n+3}, I_n \to I_{11n+4+3}$	$\langle I_7 \rangle$
<11,8,0>	$T_n \to T_{11n}, M_n \to M_{11n+8}, MI_n \to MI_{11n+0}, I_n \to I_{11n+8+0}$	$\langle I_8 \rangle$

<11,0,9>	$T_n \to T_{11n}$, $M_n \to M_{11n+0}$, $MI_n \to MI_{11n+9}$, $I_n \to I_{11n+0+9}$	$\langle I_9 \rangle$
<11,4,6>	$T_n \to T_{11n}$, $M_n \to M_{11n+4}$, $MI_n \to MI_{11n+6}$, $I_n \to I_{11n+4+6}$	$\langle I_{10} \rangle$
<11,8,3>	$T_n \to T_{11n}$, $M_n \to M_{11n+8}$, $MI_n \to MI_{11n+3}$, $I_n \to I_{11n+8+3}$	$\langle I_{11} \rangle$

A.4. The members of AUT(T/M).

Action on T/M	Hyper-operator	Order
$T_n \to T_n$, $M_n \to M_{n+0}$	$\langle T_0 \rangle$	1
$T_n \to T_n$, $M_n \to M_{n+2}$	$\langle T_2 \rangle$	6
$T_n \to T_n$, $M_n \to M_{n+4}$	$\langle T_4 \rangle$	3
$T_n \to T_n$, $M_n \to M_{n+6}$	$\langle T_6 \rangle$	2
$T_n \to T_n$, $M_n \to M_{n+8}$	$\langle T_8 \rangle$	3
$T_n \to T_n$, $M_n \to M_{n+10}$	$\langle T_{10} \rangle$	6
$T_n \to T_{5n}$, $M_n \to M_{5n+0}$	$\langle M_0 \rangle$	2
$T_n \to T_{5n}$, $M_n \to M_{5n+2}$	$\langle M_2 \rangle$	2
$T_n \to T_{5n}$, $M_n \to M_{5n+4}$	$\langle M_4 \rangle$	2
$T_n \to T_{5n}$, $M_n \to M_{5n+6}$	$\langle M_6 \rangle$	2
$T_n \to T_{5n}$, $M_n \to M_{5n+8}$	$\langle M_8 \rangle$	2
$T_n \to T_{5n}$, $M_n \to M_{5n+10}$	$\langle M_{10} \rangle$	2
$T_n \to T_{7n}$, $M_n \to M_{7n+0}$	$\langle MI_0 \rangle$	2
$T_n \to T_{7n}$, $M_n \to M_{7n+2}$	$\langle MI_2 \rangle$	6
$T_n \to T_{7n}$, $M_n \to M_{7n+4}$	$\langle MI_4 \rangle$	2
$T_n \to T_{7n}$, $M_n \to M_{7n+6}$	$\langle MI_6 \rangle$	6
$T_n \to T_{7n}$, $M_n \to M_{7n+8}$	$\langle MI_8 \rangle$	2
$T_n \to T_{7n}$, $M_n \to M_{7n+10}$	$\langle MI_{10} \rangle$	6
$T_n \to T_{11n}$, $M_n \to M_{11n+0}$	$\langle I_0 \rangle$	2
$T_n \to T_{11n}$, $M_n \to M_{11n+2}$	$\langle I_2 \rangle$	2
$T_n \to T_{11n}$, $M_n \to M_{11n+4}$	$\langle I_4 \rangle$	2
$T_n \to T_{11n}$, $M_n \to M_{11n+6}$	$\langle I_6 \rangle$	2
$T_n \to T_{11n}$, $M_n \to M_{11n+8}$	$\langle I_8 \rangle$	2
$T_n \to T_{11n}$, $M_n \to M_{11n+10}$	$\langle I_{10} \rangle$	2

A.5. The members of INN(T/M).

Conjugation	Action	Order	Image in T/M
$[T_0] = [T_3] = [T_6] = [T_9]$	$T_n \rightarrow T_n, M_n \rightarrow M_{n+0}$	1	T_0
$[T_2] = [T_5] = [T_8] = [T_{11}]$	$T_n \rightarrow T_n, M_n \rightarrow M_{n+4}$	3	T_4
$[T_1] = [T_4] = [T_7] = [T_{10}]$	$T_n \rightarrow T_n, M_n \rightarrow M_{n+8}$	3	T_8
$[M_0] = [M_3] = [M_6] = [M_9]$	$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+0}$	2	M_0
$[M_2] = [M_5] = [M_8] = [M_{11}]$	$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+4}$	2	M_4
$[M_1] = [M_4] = [M_7] = [M_{10}]$	$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+8}$	2	M_8

A.6. The cosets of x(INN(T/M)) = (Inn(T/M))x, for all x in AUT(T/M).

Action	Order	Hyper-operator
$T_n \rightarrow T_n, M_n \rightarrow M_{n+0}$	1	$<T_0>$
$T_n \rightarrow T_n, M_n \rightarrow M_{n+4}$	3	$<T_4>$
$T_n \rightarrow T_n, M_n \rightarrow M_{n+8}$	3	$<T_8>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+0}$	2	$<M_0>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+4}$	2	$<M_4>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+8}$	2	$<M_8>$

Action	Order	Hyper-operator
$T_n \rightarrow T_n, M_n \rightarrow M_{n+2}$	6	$<T_2>$
$T_n \rightarrow T_n, M_n \rightarrow M_{n+6}$	2	$<T_6>$
$T_n \rightarrow T_n, M_n \rightarrow M_{n+10}$	6	$<T_{10}>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+2}$	2	$<M_2>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+6}$	2	$<M_6>$
$T_n \rightarrow T_{5n}, M_n \rightarrow M_{5n+10}$	2	$<M_{10}>$

Action	Order	Hyper-operator
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+0}$	2	$<MI_0>$
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+4}$	6	$<MI_4>$
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+8}$	6	$<MI_8>$
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+0}$	2	$<I_0>$
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+4}$	2	$<I_4>$
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+8}$	2	$<I_8>$

Action	Order	Hyper-operator
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+2}$	6	<MI_2>
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+6}$	2	<MI_6>
$T_n \rightarrow T_{7n}, M_n \rightarrow M_{7n+10}$	6	<MI_{10}>
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+2}$	2	<I_2>
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+6}$	2	<I_6>
$T_n \rightarrow T_{11n}, M_n \rightarrow M_{11n+10}$	2	<I_{10}>