Generalized Musical Intervals and Transformations

by David Lewin

Reviewed by Bo Alphonce

When a listener wants to put analytical observations into words, common-sense language *can* be sufficient; in music analytical discourse it is used extensively. But ordinary language tends to hide rather than reveal the formal structure of a problem or an argument. And to the extent that meanings and their contexts undergo gradual change, ordinary language becomes less useful when precision and generality are essential, as they are in formal theories and should be in music analysis. The common remedy, of course, is to use the special language associated with an abstract model where explicit definitions have to be given and stringent logic must govern theorems and their interconnections.

On first contact with the intellectual demands of an abstract model of music its relation to the listener's intuitive analysis may seem remote. In that encounter it is easy to forget that the model was first constructed precisely to capture the analytic intuitions of the theorist *as listener*. If the level of abstraction is one aspect of the difficulties, another may be that listener's ability to hear and intuit musical structure at a level others may find hard to match and reproduce. The desire to capture exceptional musical sensitivity in a theoretical framework once led to the Schenkerian model. Many recent mathematical models also originated in listener intuitions, e.g. Babbitt's theorems on serialism, Forte's set theory, or Perle's theory of twelve-tone tonality. This is eminently true also of the models David Lewin discusses in his *Generalized Musical Intervals and Transformations*, a book that collects and coordinates ideas he has been developing in essays spanning three decades of theoretical exploration.

The mathematical group theory mostly implicit in the writings of Babbitt, Forte, Perle, and their students[1] is used explicitly by Lewin. This is done for the sake of generality: in order for the theory to be proven functional for a variety of musical systems besides twelve-tone pitch or pitch-class systems, its theorems have to be recognized from case to case. Or, in different terms: twelve-tone literature can allow abstract group theory to recede into the background and be felt primarily through its effect on analytical decisions; however, when the same mathematical theory serves as a common basis for widely different analytical models, it has to come to the foreground in order for the shared structure of the models to be visible.

As a result, the book spans a tightrope between two extremes: at the one end mathematical language with definitions, theorems, and proofs; at the other end musical language with score excerpts, analytic discoveries and, indeed, ear-openers. Whether dancing, walking, or struggling for balance, the reader will commute between these points throughout the book. The baggage of the

[1] It often surfaces, of course. For an example, see John Rothgeb, "Some Ordering Relationships in the Twelve-Tone System." *Journal of Music Theory* vol. 11:2 (1967), 176-97.

reader will determine which end point is home. Mathematicians and music theorists do intersect to some degree, but few mathematicians had music analysis on their arts schedule, and few music theorists had group theory in their math curriculum. For the benefit of those who (like myself) go to work at the math end Lewin has provided a first chapter of mathematical preliminaries that offers a quick introduction to those aspects of group theory that are essential in the rest of the book. For all its brevity the chapter is lucid and comprehensive; for all its lucidity, however, it runs a bit fast: the reader may find it useful to have a set- and group-oriented algebra text at hand.

Generality,.of course, is a fundamental prerequisite for any successful theory; generalization of theory in the sense of widening its scope, applicability, and flexibility, is Lewin's main concern. Chapters two through four reshape a number of more or less fragmentary segments of traditional theory into well-defined instances of his abstract Generalized Interval System (GIS) model. Chapters five and six apply the GIS to a generalization of musical set theory, and the final four chapters bring about further expansion of scope by shifting the focus from intervals to transformations, particularly as modelled by graphs and networks.

The purpose of seeking greater generality and flexibility is not, of course, to build theory for theory's sake. The overriding concern is, first, to make analytical theory come as close to musical processes as analytical intuition can possibly guide it; second, to have theory by virtue of its inner consistency and explanatory power return its own suggestions, lead the analyst to further discoveries.

This interaction between theory formation and analytic returns is demonstrated over and over in the book, often brilliantly.

The first section of the book, the one that defines and begins to explore the GIS model, leads up to a distinction which is crucial to the generality of the GIS model. Since I find a problem in the order of presentation of this material, I shall try to summarize it, just enough to make my point. In so doing, I am purposely using loose and informal language; therefore, my summary will do no justice at all to the precision of Lewin's text, nor to its inimitable flavor of friendly professorial teasing.

In brief, a GIS consists of a musical space and intervals between elements of that musical space. Chapter two offers a dozen examples of musical spaces, ranging from diatonic and chromatic pitches and their corresponding pitch-class systems, over a harmonic space in just intonation and its pitch-class analog, to various timepoint and duration spaces. (In the latter category, some examples apply directly to complex rhythmical structures found in works by Babbitt, Carter, Ligeti, and Stockhausen.) In the formal definition of a GIS, intervals make up a mathematical group (IVLS) whose binary operation is composition of intervals. The function int(s,t), where s and t are elements of the space S, maps the Cartesian product S x S into IVLS. Other musically relevant transformations are modelled by different functions.

Chapter three explores a number of GIS theorems and proves them with great elegance. These include the label function and several operations that Lewin has discussed earlier.[2] They also

[2]Several of the concepts in chapter three are prepared in "A Label-Free Development for 12-Pitch-Class Systems." *Journal of Music Theory* 21:1 (1977), 29-48.

demonstrate how a given GIS can give rise to new ones by the definition of equivalence relations partitioning its musical space into equivalence classes (as when a pitch space is partitioned into pitch classes), or by combination with another GIS. Studying Lewin's example of the latter the reader may experience mild frustration; I will return to this point, but my complaint is about something else.

In the course of the chapter Lewin devotes an important discussion to the generalization of transposition and inversion operations and their interactions. In this connection he introduces a distinction between transposition operations and interval-preserving operations. This in turn hinges on the crucial mathematical dichotomy between commutative and non-commutative groups. The problem, as we shall see, is that the reader is thrown into this discussion unprepared and unsuspecting. (Along with mounting impatience, however, one begins to wonder whether this is planned: Lewin is a shrewd pedagogue who may actually *want* the reader bewildered in order to make the denouement that much more effective.)

A GIS is commutative if its group is commutative, i.e. if the binary group operation[3] is commutative. In the group IVLS of a pitch or pitch-class GIS, composition of intervals is additive; since addition is commutative, so is the group of intervals and thus the GIS. In fact, all exemplified GISs through chapter three are commutative by virtue of either additive or multiplicative group operation; this is so for the simple reason that non-commutative

[3] Lewin uses the term "binary composition" rather than "binary operation" while some texts reserve "composition" for the binary operation on groups of operations. Having said this, I am adopting his term "composition of intervals."

interval groups do not easily suggest themselves as models of music. The one Lewin is going to present in chapter four is so important that it makes perfect sense to build toward it as a climax--but this does not help our unsuspecting reader.

Normally, transposition operations are taken for granted as interval-preserving. In non-commutative groups they are not necessarily so; therefore, the distinction is necessary. In preparation, though, Lewin has had to take his mathematical terminology as far as through anti-isomorphism, and the algebraic formulas have had to maintain strict difference between left- and right-multiplication. Meanwhile, chapter one has not sufficiently forewarned the reader about treating algebraic multiplication as non-commutative, and chapter three has only just hinted at the topic of commutativity when, suddenly, transposition and interval-preserving operations are separated. The reasons for the surprising complexities become clear in the next chapter, and it turns out that the author did indeed plan; at the threshold to chapter four he greets you with an impish smile: "the reader may have been puzzled"! While admiring the pedagogue in all this, my point is that "puzzled" will be a euphemism to a math tenderfoot; that reader will have paid a high price in question marks and confusion. In the end, of course, confusion survived is insight gained; yet, some rearrangement may be worth considering for the next edition.

So, the example of a non-commutative GIS the reader was crying for in chapter three arrives in chapter four. It models a situation in Elliott Carter's String Quartet no. 1 where several tempi are operating simultaneously and no single time-unit can be chosen as referential beat without leading to a distorted performance. This type of rhythmic texture would create an extraordinary test situation for any analytic theory. It is proof of the power of the GIS model that its representation of the rhythmic essence of the score yields precisely that practical coaching advice which would give rise to a musically meaningful performance.

The musical space of this GIS is time spans; spans, and the interval from one span to another, are measured relative to the beat of the tempo in which they occur. Intervals in different tempi can be proportionately the same, i.e. the interval may be preserved regardless of tempo. "Transposition" of time spans--displacement by so and so many times the beat--depends on the tempo for its time unit; therefore, the same transposition of two time spans from one tempo to another need not preserve their interval; the same transposition from one tempo to another, then back to the first, will not restore the original interval.

This kind of GIS enters a level of structural complexity for which little analytic theory has been developed. It strongly suggests further exploration, both compositionally and analytically, of non-commutative systems in the pitch and rhythm domains.

Rather than proceeding along those lines, however, Lewin further expands the scope of his model by formulating (commutative) GIS models for the timbre domain. These reflect closely some of the intensive current developments in timbre experimentation among computer music composers. It is gratifying to observe this rapprochement between acoustic theory and analytic theory in a shared concern for both descriptive and manipulative versatility. Noting a difference between his GISs and Wayne

Slawson's timbral model[4] Lewin launches a methodological discussion of great principal interest. I shall return to this after commenting on the remaining two sections.

Several concepts and relations familiar to the reader from musical set theory have already been included, extended, and generalized in the first section. The second section, chapters five and six, focuses specifically on set theory generalization; it concentrates on three central functions and their interactions with a group of set-theoretic operations. The Interval Function and the Embedding Function[5] relate in different ways to a generalization of Forte's Interval Vector. The Injection Function in combination with the group of transposition and inversion operations generalizes Forte's K and Kh relations.

The Interval Function answers the question how many different ways a given interval i can be spanned between members of a set X and members of a set Y. In other words, it reveals the cardinality of a shared subset transposed at T_i. For the twelve-tone pitch-class space this means that, if results are tabulated instead of counted, repeated applications of the Interval Function for i = 0 through 11 tabulate into the P-form of the Invariance Matrix for sets X and Y.[6] When Lewin claims that this "function does not figure heavily in the standard literature of atonal set theory" (p. 89), he is

[4] *Sound Color* (Berkeley: University of California Press, 1985).

[5] Discussed earlier in "Forte's Interval Vector, My Interval Function, and Regener's Common-Note Function." *Journal of Music Theory* . 21:2 (1977), 194-237.

[6] The Embedding and Injection Functions also relate in well-defined ways to the Invariance Matrix. I devoted some discussion to this concept in *The Invariance Matrix* (Ph. D. diss., Yale Univ. 1974); it has been further developed by Robert Morris in *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven and London: Yale University Press, 1987).

right as far as its precise definition goes, but the relation it establishes belongs to the central ones in that literature.

The Embedding Function instead tells how many times a form of a set X is contained in a set Y. "Form" here depends on which operations are canonical, i.e. chosen to establish equivalence between sets. Using transposition and inversion as canonical operations on pitch-class sets, repeated application of the Embedding Function with X varying through the six two-element sets yields Forte's Interval Vector for set Y. Obviously, allowing X to run through all sets of cardinalities less than the cardinality of Y would yield a complete subset list for Y.

The Interval and Embedding Functions can be seen as special cases of the more general Injection Function. This function answers the question how many elements of a set X are members of a set Y under a given transformation. The transformation need not be canonical operation, nor an operation at all in the mathematical sense. This allows a remarkable flexibility and extends musical set theory beyond the constraints of the mathematical group model. In a series of examples chapter six demonstrates both the analytical adaptability of the new construct to situations where a GIS is not defined and its power to further generalize already presented concepts. The chapter ends by generalizing the Injection Function itself into the domain of visual arts.

While all this abstraction has a nice ring to it for set theory fans (count me in), the wider appeal lies in the musical analyses strewn into the text. Lewin's ability to lead his reader up to a striking analytic discovery is not news, but to get such a fine collection of them in one place is a reward. In particular, his use of the "if only" situation--where a theory-derived explanation *almost* fits--brings analytic theory into close contact with artistic play and within reach of semantic interpretation.

The final section shifts the focus away from the idea of the interval, i.e. from the result of a transformation, to the transformation itself. This suggests a move from music as object to music as process, from observation to participation. It seems that Lewin's claim here that "we tend to conceive the primary objects in our musical spaces as atomic individual 'elements' rather than contextually articulated phenomena" (pp. 158-59) must refer in the first place to traditional theory rather than to listening attitudes. With all the play on expectations and contextual projections certainly Beethoven (Haydn, Bach, ...) required the listener to participate in the process of the music's coming into being. The objects congeal afterwards; the score with its atomic signs is outside; afterwards and outside is when and where theory was formulated. It is only recording technology that has made listening available for intensive study; meaningful listening probably always involved a "transformational attitude."

In the GIS context the shift means replacing the concept of a GIS structure by that of a group of operations on a musical space. Each construct can be generated from the other, and both are subsumable under a more general theory of transformations. As versatile tools for representing transformational analysis Lewin introduces graphs and networks; their formal definitions and theorems are presented with great clarity in chapter nine. Informally, they are used in chapters seven and eight for, among other provocative analyses, uncovering substructural motives in

Wagner's *Parsifal* and harmonic parallels in Beethoven's First Symphony.

The question of harmonic transformations brings up a discussion of Hugo Riemann's harmonic theory where I find myself in disagreement with Lewin's assessment. He suggests that Riemann "never quite worked through in his own mind the *transformational* character of his theories" and that he conceived "'dominant' and the like as *labels for Klangs* in a key, rather than as *labels for transformations*." In my picture of Riemann's harmonic theory[7] "dominant" certainly does not label a *Klang*; instead it labels a function. In Riemann's terminology the meaning of "Funktion" may come closer to a relation than to a transformation, but to the best of my understanding it comprises the dual aspect of being derived from, and pointing back to, (in this case) the tonic. Function labelling can be further exemplified by the fact that, for instance, the a minor triad ("^{o}e" in Riemann's nomenclature) assumes one of two alternative functions in the key of C major, depending on whether the harmonic context makes it substitute for the tonic of the subdominant. That is to say, it is not the *Unterklang* ^{o}e that is labelled but its relations within the key-defining network of harmonic functions.

At the same time as the final section brings the reader to a level where theoretical scope and flexibility are growing with every step, the sheer intellectual difficulties become less severe. This reflects the fact that the theory now has been developed to a point

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[7]Pieced together more than forty years ago in private studies at Uppsala, Sweden, with one of Riemann's most enthusiastic followers, Sven E. Svensson, who published a unique extension of Riemann's dualist harmonic theory in his Harmonilara (Uppsala: Almqvist & Wiksell, 1933), co-authored with Carl-Allan Moberg.

where its proximity to the listening situation is easy to demonstrate. This is borne out by the final four analyses, contained in chapter ten, of excerpts from Mozart's G-minor Symphony, K. 550, Bartok's "Syncopation" from *Mikrokosmos*, vol. 5, the first of Prokofieff's *Melodies* op. 35, and Debussy's *Reflets dans l'eau*. The last of these will lead me into some more general comments.

The Debussy analysis comprises the major portion of the piece and contains some exquisite suggestions for performance. From the theoretic point of view its various networks imply several different GIS structures; that is, different pitch organizations co-exist in the piece. From the point where analytical theory is sensitive to multiple spaces in the same composition to the point where it also handles alternative analyses concurrently there is no wide leap. This emphasis on process ties in with the phenomenological approach in one of Lewin's recent essays.[8] It also brings us back to a couple of points that were left hanging earlier in this review. Both have to do with the question to what extent analytical significance is determined by the model.

The very first analysis in the book occurs in chapter three when the theoretical development has reached the idea of a "direct-product" GIS. It explores an excerpt from the opening of the third movement of Webern's Piano Variations Op. 27. For each pair of notes, the particular GIS used here measures the distance of the second note from the first by a time-point interval and a pitch-class interval. Getting this far has taken a fair amount of theoretical apparatus; yet, what this GIS provides is a standard procedure in

[8]"Music Theory, Phenomenology, and Modes of Perception." *Music Perception* 3:4 (1986), 327-92.

atonal analysis. The cream of the analysis, the important decisions, are altogether based on criteria from outside the model, emanating from the experience and expertise of the analyst. These two factors may trigger the "mild frustration" I referred to earlier; it is like climbing the hill only to receive what one is already carrying in one's rucksack. Before dealing with the frustration, let me take a somewhat different example.

In several of his analyses Lewin makes effective use of what he calls the RICH function: chaining of retrograde inversions with an overlap of one or two elements. The discussion in chapter ten of a passage from the beginning of the development section in the finale of Mozart's G-Minor Symphony, K. 550, offers an especially powerful demonstration of the RICH-transforms involving both pitch and duration motives. The analysis is entirely convincing, but it also raises a question: can criteria for analytic significance be built into a model-based theory, or is the model always embedded in a (possibly not formulated) wider theoretic framework to which such criteria belong? It would seem trivial to analyze an equidistant scale passage as a RI-chain (or--if diatonic organization is regarded as equivalent to equidistance--any scale passage). Any four-note motive where the last two notes are a transposition of the first two is a RI-chain; how does the theory decide whether this is a non-trivial analysis?

Obviously, the frustration about the Webern example was based on the wrong expectations. For any analytic situation the model construct brings together a piece of musical structure and a set of system properties. In the Webern case, the specific GIS was triggered by an intuition that its particular properties would highlight relations the analyst would find interesting. The formal apparatus guaranteed consistency; criteria for significance remained extrinsic.

Similarly, the RICH transformation highlighted relations that emerged as significant only when viewed together and in a larger context in the Mozart excerpt. To my knowledge, there does not exist a formal theory that determines analytic significance; in this area, agreements and disagreements are based on shared musical experience and heritage, and very much on intuitive responses. An analytic decision, therefore, often has the character of *argumentum ad hominem*. "This is so" means "This is so, don't you agree?" There is nothing negative about this; music theory debate would not be very exciting if there were such things as automatic answers. Still, the question of how close formal theory can get to the intuitive realm *is* interesting. Artificial Intelligence research has not reached the border some phenomenologists would claim to be uncrossable. Not yet. In Lewin's analyses abstract models interact closely with intuition, but in all of them there is a separate level, outside of the models, where decisions are made.

The questions are amplified when the model does not closely approximate analytic intuition. Lewin finds his two timbral GIS structures wanting in this respect. In either one, he says (p. 85), "we may have $int(s_1,t_1) = int(s_2,t_2)$, while the intuitive proportion between s_1 and t_1 does not much 'sound like' the intuitive proportion between s_2 and t_2." He refers to the general problem of complex relationships between physical and perceptual measurements, e.g. between levels of amplitude and loudness in different parts of the frequency range. While one radical way out is to maintain a strict distinction between physical and phenomenal terminologies,[9] Lewin suggests flexibility in this problem area, too: "It is unfair to demand of a musical theory that it always address our sonic intuitions faithfully in all potentially musical contexts under all circumstances. It is enough to ask that the theory do so in a sufficient number of contexts and circumstances. Perhaps, too, it is fair to ask that the theory be *potentially* able to address our intuitions..." One could add that as long as analytical decisions are *supported* by but not *determined* by the theoretical model, some "intuition-deficiency" need not be critical. I cannot resist quoting Lewin's summary of the discussion: "One should not ask of a theory, that every formally true statement it can make about musical events be a perception-statement. One can only demand that a preponderance of its true statements be *potentially* meaningful in sufficiently developed and extended perceptual contexts."

A mundane matter before closing the review: The book has its fair share of typographical errors that will not bother anybody, but there are some cases where a correction might save a reader some question marks. I have collected these in a brief Appendix. In fairness to the publisher, then, it must be stressed that the complicated formula language has been rendered with exemplary precision.

In this brief discussion of Lewin's book I have fallen far short of demonstrating the full width of its musical insight and theoretical creativity. Abundant in both, it sends ideas out in a multitude of directions. There are sudden glimpses of an erudition

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[9] As suggested by Ingmar Bengtsson in his article "On Relationships Between Tonal and Rhythmic Structures in Western Multipart Music." *Svensk Tidskrift for Musikforskning* 43 (1961), 49-76.

that one would like to know much more of. But it is not easy reading. Strict speed limits are imposed on sections where the theory is developed. Without penetrating theorems and proofs step by step the reader will miss the full flavor of the analyses--and still enjoy them. There is much to be said, though, for spending all the mental energy the book requires. It is invigorating, even rejuvenating. Then there is the sense of humour just below the surface, hardly ever expressed and yet somehow always present. And, the same book that makes frequent use of "anti-homomorphism" also contains words like "potato chips," "football," "can of beer." Happy reading!

Appendix

Page 39, line 5: "C-C#" should read "C#-C"

Page 55, line 2 from below: "inversion-preserving" should read "interval preserving"

Page 56, section 3.5.8, line 3 of Proof: "j" missing from end of right side expression

Page 142, line 10: "INJ(X,Y) (f)" should read "INJ(Y,Y) (f)"

Page 145, line "(C)" under Proofs: second "=" should read "-"

Page 179, Fig. 8.2 a) "#" missing after "G" in "(G ,-)"

Page 232, Fig. 10.10: Check your score. Measures in mine[10] are ahead by 1 from m. 24 on.

[10] The Nov. 1955 printing of the 1905 Durand & Fils edition.