Some Compositional and Analytic Applications of T-Matrices

by

Robert Morris

Much is already known about the row table, "that 12 X 12 Latin Square, wherein so many good musical things reside."[1] Not only does the table list the 48 forms in a row-class[2] in the smallest possible space, the table's layout illustrates general twelve-tone invariances possessed by all possible rows. In addition, for those well-versed in the mysteries of "invariance matrices," the table allows one to glean much about the particular structure of its generating row.[3] In this paper, I will confine myself to the T-matrix, of which the row table is an instance. After reviewing the main properties of such tables and matrices, I shall present a few new--or at least unpublished--features and applications of T-matrices, among them the embedding of one ordered pc segment (henceforth, *pcseg*) in

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[1]Milton Babbitt, "Responses: a First Approximation," *Perspectives of New Music* 14/2 and 15/1 (double issue, 1973-4):4.

[2]A row-class is the familiar set of 48 transforms of a twelve-tone row. For other basic definitions used in this article see the glossary in Robert Morris, *Composition with Pitch-Classes: A Theory of Compositional Design* (New Haven: Yale University Press, 1987).

[3]The basic theory of invariance matrices is found in Bo Alphonce, "The Invariance Matrix" (Ph.D dissertation, Yale University, 1973); and Carlton Gamer and Paul Lansky, "Fanfares for the Common Tone," *Perspectives of New Music* 14/2 and 15/1 (double issue, 1973-4):128-40. See Morris, *Composition* . . . for other interpretations of such matrices in pitch, temporal, and transformational "spaces."

another, and the polyphonic alignment of two related or independent pcsegs.

Basic Properties of T-Matrices and Their Interpretations

I begin with a row table for the twelve-tone row P, <014875936AB2>. Its table is given in example 1. The table forms a "latin square" since each element (i.e., each pc) of P is found once and only once in every row and column in the square.

It is well known that a row table shows its prime row, P, as its top row (read from left to right), with the transpositions of P in the other rows of the table, arranged in order (from top to bottom) according to the inversion of P, T_0IP. Similarly, the transpositions of the inversion of P are found in the columns of the table (read top to bottom), arranged in order from left to right according to P itself. (The R and RI rows are found by reading the P and I rows backwards.) This symmetric disposition of P and I row transforms enforces an inversional symmetry around the table's "main diagonal"--the diagonal of elements from the upper left-hand to the lower right-hand corner of the table--that results in the familiar series of zeros that diagonally bisects the table. More generally, any pc x found in row a and column b will have its inversion, 12 -x(mod-12), located in row b and column a.

The main diagonal is by no means the only diagonal to have strucural importance, however. For instance, the 11-element diagonal to the immediate right of the main diagonal gives the "INT,"

Example 1: Row table for P, <014875936AB2>

0	1	4	8	7	5	9	3	6	A	B	2
B	0	3	7	6	4	8	2	5	9	A	1
8	9	0	4	3	1	5	B	2	6	7	A
4	5	8	0	B	9	1	7	A	2	3	6
5	6	9	1	0	A	2	8	B	3	4	7
7	8	B	3	2	0	4	A	1	5	6	9
3	4	7	B	A	8	0	6	9	1	2	5
9	A	1	5	4	2	6	0	3	7	8	B
6	7	A	2	1	B	3	9	0	4	5	8
2	3	6	A	9	7	B	5	8	0	1	4
1	2	5	9	8	6	A	4	7	B	0	3
A	B	2	6	5	3	7	1	4	8	9	0

the series of adjacent directed pc-intervals of the generating row.[4] In this case, the INT of P, <134BA463413>, is found as promised on the diagonal to the right of the zeros.[5] At this point, rather than continue to examine the meaning of the other diagonals on the table, let us instead look at the table from another perspective, one which explains, among other things, why the INT of P appears on the row-table generated by P.

As asserted above, the row table is a species of T-matrix which, in turn, can be described as a listing of the pc-intervals from one ordered set of pcs to another.[6] Now an ordered pc-interval is defined from pc x to y as (y-x), taken mod-12. Thus, the T-matrix E lists the pc-intervals from the pcs in set X to those in set Y such that the m^{th} pc in X and the n^{th} pc in Y form an interval that is placed in

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[4] Any ordered set of pitch-classes (pcseg) X sets up a series of (directed) pc-intervals called the INT of X, or INT(X). The interval between the n^{th} and $(n+1)^{th}$ pc in X is found as the n^{th} interval in INT(X). Thus, if X is <abcd>, then INT(X) is <b-a, c-b, d-c>. INTs have a number of properties discussed in detail in Morris, *Composition* . . . . I provide only a few here: INT(X) = INT(T_nX); INT(T_nIX) = I(INT(X)); INT(RT_nX) = RI(INT(X)); INT(RT_nIX) = R(INT(X)). The transposition of INT(X), T_n(INT(X)) has no utility in the context of this paper. We can also define $INT_m(X)$ such that the interval from X_n to $X_{<n+m>}$ is in $INT_m(X)$. INT(X) is the same as $INT_1(X)$; $INT_0(X)$ is a series of zeros; $INT_2(X)$ provides the succession of intervals between pcs separated by 1 pc in X; and so forth.

[5] This row property and others, some of which we discuss below, are found in Babbitt, "Since Schönberg," *Perspectives of New Music* 14/2 and 15/1 (double issue, 1973-74):3-28.

[6] T-matrices for unordered sets are also possible. They list the pc-intervals from one unordered pcset X to another Y. If X is the same as Y, then the matrix can be interpreted as containing the unordered pc-interval (interval-class) within X itself. Morris, *Composition* . . . (pp. 38-9,67-9) defines the ic-content of a pcset X as the content of a T-matrix made from X and itself.

the m^{th} row and the n^{th} column of E. In algebraic form this becomes:

$$E_{<m,n>} = Y_n - X_m.$$

Example 2 shows a small, rectangular T-matrix for the sets X = {023} and Y = {24}. For instance, the interval from X_1 to Y_0 is Y_0-X_1 = 2-2 = 0, which is in the intersection of row 1 and column 0 in the matrix. Note that example 2 aligns the X and Y sets to the left and over the matrix, respectively. Thus, the pcs of X and Y actually intersect in the matrix to form their pc-intervals. Now let us consider what happens when the sets X and Y that generate a T-matrix are the same. The matrix gives the pc-intervals from X to itself. If X is an ordered set, then the matrix can be interpreted as a list of the pc-intervals between the elements of X. (Where X is a twelve-tone row, the T-matrix is a row table.) It is important to note that if X is not a complete aggregate, then the T-matrix will not list all transpositions and inversions of X.

Example 2: T-matrix for {023} and {24}

	2	4
0	2	4
2	0	2
3	B	1

Returning to the diagonals on the row table of P, we can see that the main diagonal of the T-matrix (row table) contains the intervals from each element of P to itself: the intervals from P_n to P_n, each of which is 0, for all n. The intervals for adjacent elements of P, from P_n to P_{n+1}, are given by the positions intersecting the n^{th} rows and $(n+1)^{th}$ columns (that is, in $E_{<n,n+1>}$). These positions form the diagonal to the right of the main diagonal. This is why INT(P) can be read from the matrix. Moreover, the next diagonal of E comprises the positions from intersections of row n with column n+2, thus giving the interval from P_n to P_{n+2}, or the interval between successive pcs in P separated by one intervening pc. This 10-pc series <47392A9754> is the $INT_2(P)$. The other diagonals on the T-matrix are likewise related.[7]

Considering the row table of P as the intervals from P to a copy of P provides another interpretation of the diagonals. Here, the diagonals give the relation of P to its copy via "shifting."[8] Example 3a shows such correspondences. The main diagonal is the relations of intervals of P to P, pc to pc. The diagonal to the right gives the intervals from P to P shifted by one pc; the next diagonal provides the intervals from P to P shifted by two pcs; and so on. Thus the diagonals give the vertical intervals between canonic presentations of P.

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[7]For further discussion see Babbitt, "Since Schönberg," and Morris *Composition* . . . (pp.40, 107-9).

[8]The use of T-matrices to deal with canonic relations between pcsegs is a subset of the theory of rotational arrays as found in Morris, "Generalizing Rotational Arrays," *Journal of Music Theory* 32/1 (1988):75-132. This article cites the important earlier articles on such arrays, such as John Rogers, "Toward a System of Rotational Arrays," *Proceedings of the American Society of American Composers* 2 (1967):61-122.

Example 3: Interpretations of Diagonals of P

Ex. 3a:

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         P:   0 1 4 8 7 5 9 3 6 A B 2
         P:   0 1 4 8 7 5 9 3 6 A B 2

 intervals:   0 0 0 0 0 0 0 0 0 0 0 0


         P:   0 1 4 8 7 5 9 3 6 A B 2
         P:     0 1 4 8 7 5 9 3 6 A B 2

 intervals:     1 3 4 B A 4 6 3 4 1 3


         P:   0 1 4 8 7 5 9 3 6 A B 2
         P:       0 1 4 8 7 5 9 3 6 A B 2

 intervals:       4 7 3 9 2 A 9 7 5 4
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Ex. 3b:

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         P:   0 1 4 8 7 5 9 3 6 A B 2
        RP:   2 B A 6 3 9 5 7 8 4 1 0

 intervals:   A 2 6 2 4 8 4 8 A 6 A 2


         P:   0 1 4 8 7 5 9 3 6 A B 2
        RP:     2 B A 6 3 9 5 7 8 4 1 0

 intervals:     B 5 A 1 2 0 A B 2 7 1

         P:   0 1 4 8 7 5 9 3 6 A B 2
        RP:       2 B A 6 3 9 5 7 8 4 1 0

 intervals:       2 9 9 B 6 6 1 3 3 A
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But what of the diagonals that are perpendicular to the main diagonal? The basic perpendicular diagonal, called the "secondary diagonal," is the sole 12-element span from the lower left to the upper right of the matrix. For our P it is <A2624848A6A2>. The secondary diagonal gives the intervals from P_n to $P_{B\text{-}n(mod12)}$, the intervals formed when P is aligned with its retrograde. And like the situation above, the next diagonal to the right gives the intervals from $P_{B\text{-}n}$ to P_{n+1}, or from the retrograde of P to P shifted to the left by one pc; and so on for the other diagonals. Example 3b shows such alignments.[9]

It should also be understood that each of the diagonals parallel to the secondary diagonal are RI-symmetric due to the I symmetry of a T-matrix (row table) generated from one set. Another way to understand the symmetry follows from the alignment of an ordered set and its retrograde. Such an alignment produces a series of vertical intervals from pc a in P to pc b in RP followed by the retrograde of the inversion of those intervals, namely the vertical intervals from pc b in P to pc a in RP. The same symmetry obliges rows that are RT_6 invariant to display 6s on the secondary diagonal of their rows.

Segmental Inclusion and T-Matrices

In twelve-tone analysis or composition it is often very useful to know if a given pcseg is embedded in a row and/or its transforms.

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[9]This means that the pc-intervals between a pcseg and its retrograde can generate rotational arrays analogous to the "Stravinskian" type based on the INTs of a pcseg. Each T-matrix diagonal parallel to the secondary diagonal (wrapping around the matrix) will be a column in the rotational array.

(An embedded pcseg is included in some row in order, but not necessarily adjacently.) The pcseg may consist of certain pcs of the row that are highlighted to form a figure/ground presentation of the row (i.e., the segment is figure, the rest of the row is ground, or vice versa). Of course, if we wish to find a specific segment embedded in a row, we simply look. However, if we want to see if a segment Z is embedded in *any* row in the row-class of P, looking at each and every row in the row table is tedious and time-consuming.[10] Another, more sophisticated method would be to generate the T- and I-matrices for the segments Z and P. Although this method is valuable and highly general, producing the matrices and inspecting them for certain patterns of identical integers requires pencil and paper (or an eidetic memory). But before providing a much simpler hand-algorithm performable on the row-table itself, let us consider example 4. The left side of the example shows the six distinct ways the pcseg Z <0152> is embedded in members of the row-class of P. The right half shows the isomorphic situation, the cases of distinct serial transforms[11] of Z embedded in the row P. On the left, Z is included in GP (where G is a serial operation); on the right, HZ is embedded in P (where H is the inverse of G).

Our algorithm generates the right half of example 4, indicating which transforms of Z, if any, are embedded in P. Only the INT of Z and the row table of P are needed to perform it. In general, we shall be attempting to find a zigzag sequence through the

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[10]One can also use an "order number table" as shown in Morris, *Composition . . .*, p.115-16.

[11]"Serial transforms" or "serial operators" are members of the set of Tn, I, R, and/or r_n (rotation) operators commonly used in serial and twelve-tone theory.

Example 4: Embeddings of pcset Z in row P

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P:       014875936AB2      P:        014875936AB2
Z:       01   5     2      Z:        01   5     2

T_8:     8904315B267A      P:        014875936AB2
Z:         0  15 2         T_4Z:       4  59 6

RT_A:    0984173562BA      P:        014875936AB2
Z:       0   1  5 2        RT_2Z:      4 7  3   2

T_4I:    43089B71A652      P:        014875936AB2
Z:         0    1  52      T_8IZ:       87  36

RT_AIP:  8B047153269A      P:        014875936AB2
Z:         0  15 2         RT_AIZ:      8 59  A

RT_BIP:  9015826437AB      P:        014875936AB2
Z:        015 2            RT_BIZ:         9 6AB
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Example 5:

0	b	c	d	e	f	g	h	i	j	k	l
-b	0	.	.	.	.	x					
-c		0				.					
-d			0			.					
-e				0		.					
-f					0	.					
-g						0	.	x			
-h							0	.			
-i								0	x		
-j									0		
-k										0	
-l											0

table and the positions on the main diagonal. Example 5 shows the form of such a sequence in an "abstract" row table.

All entries in the row table of example 5 are blanked out except: (1) the top row which contains the row P, denoted by the string <0bcdefghijkl>; (2) the elements of IP in the zeroth column <0 -b -c -d -e ... -k -l>; (3) the main diagonal of zeros; and (4) the sequence itself, consisting of the zeros of the main diagonal alternating with the non-zero[12] entries given by xs. The dots merely indicate entries between those on the sequence. The matrix positions traversed by the sequence in example 5 are $0 = E_{<1,1>}$, $x = E_{<1,6>}$, $0=E_{<6,6>}$, $x = E_{<6,8>}$, $O = E_{<8,8>}$, $x = E_{<8,9>}$, $O = E_{<9,9>}$. Obviously, the algorithm depends on the values of the xs, the non-zero entries in the sequence. These are the successive intervals in the INT of Z. As a result, if there is a sequence on the table using all the values of INT(Z), then Z or a transposition of Z is embedded in P (or Z is embedded in P or a transposition of P). A nice feature of the algorithm is that one can read the transform of Z from the table by taking those entries in the top row of the table that are over--in the same column as--the zeros of the sequence. (Thus, in example 5, <bgij> would be a transposition of Z).

Let us use the algorithm to show that, for the Z and P of example 4, Z and T_4Z are included in P. The relevant sequences in the row table of P are given in example 6a, each underlined in its own copy of the row table. The INT of Z is <149>.

[12]In more general cases, where pcseg Z or P has pc duplication, zeros will appear off the main diagonal.

In the left table of example 6a, we begin with the 0 in $E_{<0,0>}$ and find the first element of INT(Z), 1, in $E_{<0,1>}$. Then, after dropping down to the main diagonal to $E_{<1,1>}$, we look to the right on row 1 and find the next value of INT(Z), 4, in $E_{<1,5>}$. (If we were unable to find a 4 on row 1, there would be no transposition of Z in P that began with the first pcs of P.) We now move to $E_{<5,5>}$ and again look to the right on row 5 for the last element of INT(Z), which is found in the last column of the table. Dropping down to the main diagonal leaves us in $E_{<B,B>}$. Looking on the top row in the columns that have zeros from the sequence gives us the positions $E_{<0,0>}$, $E_{<0,1>}$, $E_{<0,5>}$, and $E_{<0,B>}$ which holds the T_0 of Z (i.e., Z itself), <0152>. A similar sequential construction is shown in the table on the right side of example 6a. We shall start with row 1 of the table since we have already found an embedding of Z starting with row 0. There the first element of INT(Z), 1, is found on row 1 of the table, but unfortunately in the last column, so a sequence cannot be extended further. Looking on row 2, we find the requisite 1 in $E_{<2,5>}$. This begins a sequence that proves to be completed. The main diagonal elements are $E_{<2,2>}$, $E_{<5,5>}$, $E_{<6,6>}$, and $E_{<8,6>}$, so $E_{<0,2>}$, $E_{<0,5>}$, $E_{<0,6>}$, and $E_{<0,8>}$ hold a transposition of Z, this time T_4Z.[13]

When we wish to find other transforms besides transpositions of Z in P, we simply search for the INT of RZ (for RT_nZ in P), INT(IZ) (for T_nIZ in P), and/or INT(RIZ) (for RT_nIZ in P).[14] Examples 6b and 6c show sequences for INTs of

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[13] We continue to begin with the zeros on successive rows of the matrix, but no more complete sequences are found.

[14] See previous note 4.

Example 6a:

T_0Z:	0	1				5						2
	0	1	4	8	7	5	9	3	6	A	B	2
	B	0	3	7	6	4	8	2	5	9	A	1
	8	9	0	4	3	1	5	B	2	6	7	A
	4	5	8	0	B	9	1	7	A	2	3	6
	5	6	9	1	0	A	2	8	B	3	4	7
	7	8	B	3	2	0	4	A	1	5	6	9
	3	4	7	B	A	8	0	6	9	1	2	5
	9	A	1	5	4	2	6	0	3	7	8	B
	6	7	A	2	1	B	3	9	0	4	5	8
	2	3	6	A	9	7	B	5	8	0	1	4
	1	2	5	9	8	6	A	4	7	B	0	3
	A	B	2	6	5	3	7	1	4	8	9	0

T_4Z:			4			5	9		6			
	0	1	4	8	7	5	9	3	6	A	B	2
	B	0	3	7	6	4	8	2	5	9	A	1
	8	9	0	4	3	1	5	B	2	6	7	A
	4	5	8	0	B	9	1	7	A	2	3	6
	5	6	9	1	0	A	2	8	B	3	4	7
	7	8	B	3	2	0	4	A	1	5	6	9
	3	4	7	B	A	8	0	6	9	1	2	5
	9	A	1	5	4	2	6	0	3	7	8	B
	6	7	A	2	1	B	3	9	0	4	5	8
	2	3	6	A	9	7	B	5	8	0	1	4
	1	2	5	9	8	6	A	4	7	B	0	3
	A	B	2	6	5	3	7	1	4	8	9	0

Example 6b:

RT_2Z:			4		7			3				2
	0	1	4	8	7	5	9	3	6	A	B	2
	B	0	3	7	6	4	8	2	5	9	A	1
	8	9	0	4	3	1	5	B	2	6	7	A
	4	5	8	0	B	9	1	7	A	2	3	6
	5	6	9	1	0	A	2	8	B	3	4	7
	7	8	B	3	2	0	4	A	1	5	6	9
	3	4	7	B	A	8	0	6	9	1	2	5
	9	A	1	5	4	2	6	0	3	7	8	B
	6	7	A	2	1	B	3	9	0	4	5	8
	2	3	6	A	9	7	B	5	8	0	1	4
	1	2	5	9	8	6	A	4	7	B	0	3
	A	B	2	6	5	3	7	1	4	8	9	0

Example 6c:

T_4IZ:

		4					3			B	2
0	1	4	8	7	5	9	3	6	A	B	2
B	0	3	7	6	4	8	2	5	9	A	1
8	9	0	4	3	1	5	B	2	6	7	A
4	5	8	0	B	9	1	7	A	2	3	6
5	6	9	1	0	A	2	8	B	3	4	7
7	8	B	3	2	0	4	A	1	5	6	9
3	4	7	B	A	8	0	6	9	1	2	5
9	A	1	5	4	2	6	0	3	7	8	B
6	7	A	2	1	B	3	9	0	4	5	8
2	3	6	A	9	7	B	5	8	0	1	4
1	2	5	9	8	6	A	4	7	B	0	3
A	B	2	6	5	3	7	1	4	8	9	0

T_8IZ:

			8	7			3	6			
0	1	4	8	7	5	9	3	6	A	B	2
B	0	3	7	6	4	8	2	5	9	A	1
8	9	0	4	3	1	5	B	2	6	7	A
4	5	8	0	B	9	1	7	A	2	3	6
5	6	9	1	0	A	2	8	B	3	4	7
7	8	B	3	2	0	4	A	1	5	6	9
3	4	7	B	A	8	0	6	9	1	2	5
9	A	1	5	4	2	6	0	3	7	8	B
6	7	A	2	1	B	3	9	0	4	5	8
2	3	6	A	9	7	B	5	8	0	1	4
1	2	5	9	8	6	A	4	7	B	0	3
A	B	2	6	5	3	7	1	4	8	9	0

RZ and IZ. (These INTs are <38B> and <B83>, respectively.) The reader may want to construct the sequences in the table (in example 1) that locate transpositions of RIZ in P; from example 4, we know there are two different embeddings.

The principle behind the algorithm derives from the way the intervals between elements of P are ordered in the table. If the n^{th} element of INT(Z) occurs in the table in position $E_{<a,b>}$, then that interval occurs both between Z_n and Z_{n+1} and between P_a and P_b. The next interval in INT(Z) is the $(n+1)^{th}$, which must span from P_b to P_c, where $c > b$. We can determine whether this is the case by seeing if the interval is found in any position $E_{<b,c>}$ where $c > b$.[15] If so, we can continue with the next interval in INT(Z); if not, only a portion of T_nZ is embedded in P.[16]

The algorithm is not limited to twelve-tone rows. We can look for transpositions of any Z in a pcseg S of any cardinality with or without pc duplications. We simply construct the T-matrix for S and perform the algorithm. A computer program implementing the algorithm can be constructed for Z and S or arbitrary length.[17] To

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[15]The use of position $E_{<b,b>}$ between $E_{<a,b>}$ and $E_{<b,c>}$ is actually superfluous. When performing the algorithm by hand, the "extra position" helps locate row b quickly on the matrix as well as providing a vivid sequence of right triangles each of whose hypotenuse lies on the main diagonal.

[16]Our algorithm can also be adapted to construct a pcseg S that embeds as many serial transforms of a specified pcseg Z as possible. Even so, a more direct algorithm is found on pages 140-45 of Andrew Mead, "Some Implications of the Pitch Class/Order Number Isomorphism Inherent in the Twelve-Tone System, Part One," *Perspectives of New Music* 26/2 (1988):96-163.

[17]Like the general analytic use of invariance matrices proposed in Alphonce, *The Invariance Matrix*, our algorithm can look for all occurrences of any pcseg Z (and its transpositions) embedded in any series of pcs S from any piece of music. The

this end, Appendix A presents a more formal presentation of the algorithm in "pseudo-code."

T-matrices and Pitch-Class Polyphony

The discussion of example 3 indicated that the diagonals of a T-matrix generated by a single pcseg S reveal that vertical pc-intervals that are found between the voices of a simple two-part canon based on S. We can extend the use of T-matrices to understand and control more general and flexible polyphonic relations between any two pcsegs S and S'. S' might be a serial transform of S, producing a canon by transposition, inversion and/or retrograde with S; or S' might be completely unrelated to S, merely producing polyphony. In either case, the T-matrix based on S and S' not only models the intervals between S and S' shifted in lockstep by n pcs, but for <u>all</u> cases of their rhythmic alignment.

To illustrate this considerable feature of the T-matrix, we will examine a few measures from the first movement of Bartòk's *Music for Strings, Percussion, and Celesta*. The passage is taken from the beginning of the last section of the movement, where Bartòk's famous four-phrase fugue theme is presented with itself in inversion.

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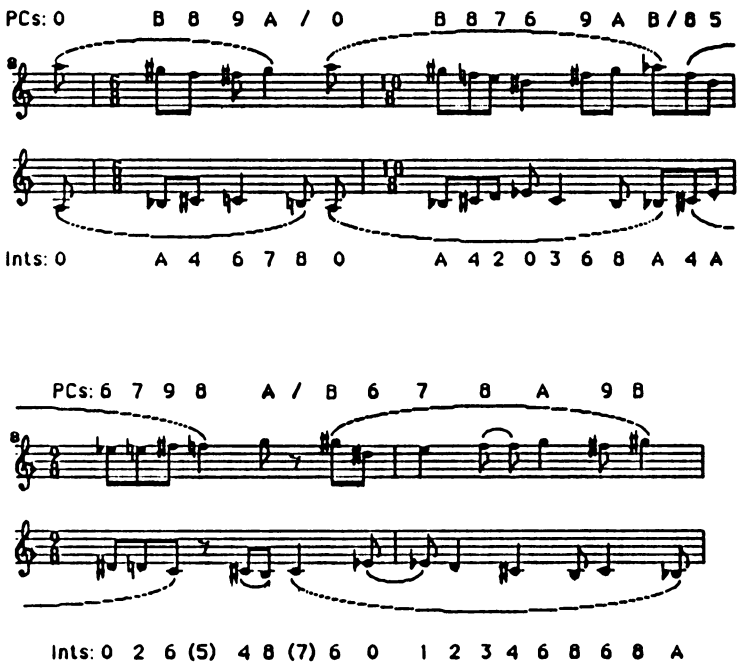
advantage of our approach over that of the general invariance matrix method is that, in the latter, a new set of matrices must be generated for each new Z, even if S remains the same. (In addition, we need only generate a little less than half the entire matrix since our algorithm only inspects positions to the right of the main diagonal.) Moreover, the scanning algorithm for detecting patterns in the general method is a good deal more complex than in our algorithm. On the other hand, the general method automatically produces important additional information about the transformations of Z vis-a-vis S.

Example 7 extracts the pair of inversionally related voices from the passage, with the original theme in the lower voice at the same pitch level as it was presented alone at the beginning of the piece. The lower voice displays the same sequence of pitches and pcs as its original statement. Only the rhythm has been altered: each of the eighth rests originally separating its four phrases has been omitted, some of the notes of a quarter-note duration have been shortened to eighths, and an eighth rest has been added after the C-natural in the third phrase. As a result, the rhythmic profile of the opening has been flattened out. The relation between the two voices is not one-to-one; rather there are slight differences in length between inversionally corresponding pcs. These occur more frequently as the passage continues, starting as deviations at the ends of the first two phrases, then involving the inserted rest (mentioned above) in the third phrase, and finally affecting almost all of the last phrase. The effect is that of the emergence of polyphony out of homorhythm analogous to two slightly differing frequencies that start in sync but gradually get out of phase.

Example 7 also lists the pc-intervals (from low to high) between the two voices. If the two voices were exactly inversionally aligned throughout, we would expect the resultant intervals to be all even (because the axis of inversion is a single pc, the index of inversion must be even). The rhythmic displacement that occurs brings in intervals of odd size, such as the 3 in the second phrase. Aside from offering some effective variety, the displacements have a motivic role. As an example, the wedge of expanding intervals in the last phrases emphasizes the procedures of expansion/contraction that mark so many aspects of the entire piece. Displacements via rests in the third phrase produce the intervals of 5 and 7 over the C-

Example 7.


PCs: 0 B 8 9 A / 0 B 8 7 6 9 A B / 8 5
Ints: 0 A 4 6 7 8 0 A 4 2 0 3 6 8 A 4 A
PCs: 6 7 9 8 A / B 6 7 8 A 9 B
Ints: 0 2 6 (5) 4 8 (7) 6 0 1 2 3 4 6 8 6 8 A

natural, and tend to suggest a mild cadence to C. This is reminiscent of the role of C in the movement as the mid-way point between the "poles" of A and E-flat, and of the minor third relation between the first and third phrases; the latter is derived via T_3 from the former.

In order to study these displacements in the Bartok example, T-matrices are generated for each phrase with A-natural as pc 0. The matrices display all of the pc-intervals from the lower voice to the higher and are therefore built out of two pcsegs. S is the lower voice, S' (equivalent to T_0IS) is the higher voice. Example 8 gives the four T-matrices. In each case, S is the vertical segment and S' is the horizontal pcseg.

If the alignment of pcs of S and S' are one to one, the vertical intervals in the polyphony would be found in the matrix's main diagonal. If S gets ahead of S' by one pc, then the interval in question is found right below those in the main diagonal. Likewise if S' gets ahead of S, the resulting intervals are found to the right of the main diagonal. Thus, moves on the matrix imply alignments between S and S': if one moves adjacently on the matrix either down or to the right or both, the resultant intervals that are traversed are the same as those made by holding or moving through the pcs of either or both of the voices S and S'. More precisely, letting Sa simultaneously sound with S'b, the interval from the former to the latter is found in $E_{<a,b>}$. There are three possible adjacent moves down and/or to the right on the T-matrix.

(1) Moving from $E_{<a,b>}$ to $E_{<a,b+1>}$, the interval in $E_{<a,b+1>}$ is that produced by sustaining Sa and moving to the next pc of S', S'_{b+1}.

Example 8: T-matrices for phrases from Bela Bartòk's
Music for Strings, Percussion, and Celesta

```
    0 B 8 9 A
--------------
0 | 0 B 8 9 A
1 | B A 7 8 9
4 | 8 7 4 5 6
3 | 9 8 5 6 7
```

phrase 1

```
    0 B 8 7 6 9 A B
--------------------
0 | 0 B 8 7 6 9 A B
1 | B A 7 6 5 8 9 A
4 | 8 7 4 3 2 5 6 7
5 | 7 6 3 2 1 4 5 6
6 | 6 5 2 1 0 3 4 5
3 | 9 8 5 4 3 6 7 8
2 | A 9 6 5 4 7 8 9
1 | B A 7 6 5 8 9 A
```

phrase 2

```
    8 5 6 7 9 8 A
------------------
4 | 4 1 2 3 5 4 6
7 | 1 A B 0 2 1 3
6 | 2 B 0 1 3 2 4
5 | 3 0 1 2 4 3 5
3 | 5 2 3 4 6 5 7
4 | 4 1 2 3 5 4 6
2 | 6 3 4 5 7 6 8
```

phrase 3

```
    9 6 7 8 A 9 B
------------------
3 | 6 3 4 5 7 6 8
6 | 3 0 1 2 4 3 5
5 | 4 1 2 3 5 4 6
4 | 5 2 3 4 6 5 7
2 | 7 4 5 6 8 7 9
3 | 6 3 4 5 7 6 8
1 | 8 5 6 7 9 8 A
```

phrase 4

(2) Moving from $E_{<a,b>}$ to $E_{<a+1,b>}$, the interval in $E_{<a+1,b>}$ is that produced by sustaining S'b and moving to the next pc in S, S_{a+1}.

(3) And moving from $E_{<a,b>}$ to $E_{<a+1,b+1>}$, the interval in $E_{<a+1,b+1>}$ is that produced by simultaneously moving to the next pcs in both S and S', that is, to S_{a+1} and S_{b+1}.

Returning to the T-matrix of phrase 1 given in example 8, note that the underlined positions are the intervals between the higher and lower voices of phrase 1 in example 7. Since the first four attacks are one-to-one correspondences of S' and S, we remain on the main diagonal. With the move of F-sharp to G in S while S' sustains C, we move to the right on the matrix, reflecting the interval of 7 between C and G. Then S' sustains while S moves to B-natural; this is shown on the matrix by moving down from $E_{<3,3>}$ to $E_{<3,4>}$. Since each of the voices has been sustained once while the other has moved to its next pc, the voices regain their inversional correspondence. Similarly, on the matrix we have moved back on to the main diagonal.

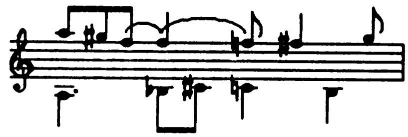
An inspection of the matrices of the other three phrases shows that the alignments remain directly inversional most of the time, save for the last phrase, which follows a zigzag path typical of a suspension-like texture. Of course, other paths and, therefore, other alignments may be traced. Example 9 gives three of these other, more deviant paths through the matrix of phrase 1, together with their pc interpretations and realizations on the staff.

An interesting theoretical question is suggested by the previous discussions, namely, how many different alignments are

Example 9.

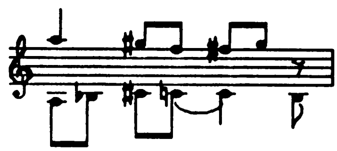
	0	B	8	9	A
0	0	B	8	9	A
1	B	A	7	8	9
4	8	7	4	5	6
3	9	8	5	6	7
2	A	9	6	7	8

S' 0 B 8 8 8 8 9 9 A
S 0 0 0 1 4 3 3 2 2
int 0 B 8 7 4 5 6 7 8



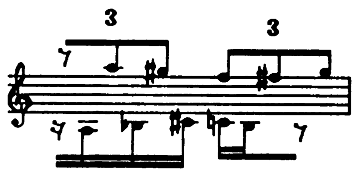
	0	B	8	9	A
0	0	B	8	9	A
1	B	A	7	8	9
4	8	7	4	5	6
3	9	8	5	6	7
2	A	9	6	7	8

S' 0 0 B 8 9 A
S 0 1 4 3 3 3 2
int 0 B 7 5 6 7



	0	B	8	9	A
0	0	B	8	9	A
1	B	A	7	8	9
4	8	7	4	5	6
3	9	8	5	6	7
2	A	9	6	7	8

S' 0 0 B B 8 8 9 A
S 0 0 1 1 4 3 2 2
int 0 B A 7 5 6 7



possible between two pcsegs S and S'? The answer is given by the following formula:[18]

$$\#\text{alignments} = \sum_{n=0,h} \frac{(g-n)!}{(e-n)!(f-n)!(n!)}$$

Where e = #S - 1; f = #S' - 1; g = e + f, and h = e or f, whichever is smaller.

For pcsegs of cardinalities 3 and 4, e = 3, f = 2, g = 5 and h = f = 2. Inserting these values in the formula shows that there are 25 distinct ways to align a segment of 3 pcs with one of 4. The following verifies this result; example 10 gives all 25 alignments grouped according to the number of intervals in each.

$$n=0 \qquad \frac{(5-0)!}{(3-0)!(2-0)!(0)!} = \frac{5!}{3!2!0!} = 10$$

$$n=1 \qquad \frac{(5-1)!}{(3-1)!(2-1)!(1)!} = \frac{4!}{2!2!1!} = 12$$

$$n=2 \qquad \frac{(5-2)!}{(3-2)!(2-2)!(2)!} = \frac{3!}{1!0!2!} = 3$$

---

grand sum: 25

[18] A proof of the formula shows that the possible alignments are isomorphic to the enumeration of possible distinct permutations of the correct number of any of the three moves (step adjacently right, step adjacently down, step adjacently right and down) through a particular T-matrix, from its upper left-hand corner to its lower right-hand corner.

Example 10: Possible alignments of segments S and S'

S = <abc>   S'= <deg>

six intervals:

abcccc dddefg	aabccc deeefg	aaabcc deffgg
abbccc ddeefg	aabbcc deeffg	aaaabc defggg
abbbcc ddeffg	aabbbc deefgg	
abbbbc ddefgg	aaabcc defffg	

five intervals:

abccc deefg	abbcc deffg	abbbc defgg	abccc ddefg	aabbc defgg	aabbc deffg
aaabc defgg	aabcc deefg	abbcc ddefg	aaabc deffg	aabbc deefg	abbbc ddefg

four intervals:

abcc defg	abbc defg	aabc defg

The variable n, dependent on h, determines how many intervals are formed by the alignment of the pcsegs. This means that each component of the sum as given above produces the number of distinct alignments of an equal number of intervals. When n = 0, the number of intervals is maximal and equal to the sum of the cardinalities of the two pcsegs . When n is maximal, equal to h, the length of the smaller of the two pcsegs minus 1, the number of intervals is minimal, equal to h plus the difference between e and f (or between the two lengths of the two pcsegs) plus 1. With the previous values, when n = 0, the number of intervals is 6 and the number of alignments is 10; when n = 1, there are 5 intervals in 12 different alignments; when n is maximally 2, there are 4 intervals in 3 different alignments.

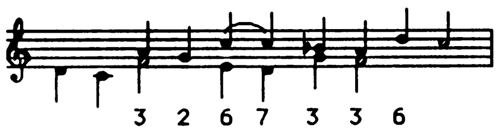
A final application of T-matrices to pc polyphony is directly related to traditional counterpoint. Taking the diatonic pc sequences <D C F F E D G F F> and its transposition a perfect fifth higher <A G C C B A D C C>, we can compose a T-matrix whose entries are the diatonic (mod-7) intervals[19] between the first and second sequences. The result is in example 11. Looking at diagonal paths of intervals, there are two that do not violate traditional dissonance treatment: <3267336> and <66656>. These are underlined in example 11a. Converting the alignments determined by the sequences gives two musical examples, examples 11b and 11c. The intervals now are bass figures. The first example describes two suspensions with a change of bass; the second example shows a simple and legal sequence of fifths and sixths. The T-

[19]These intervals are "adjusted" to conform to tradition so that the prime is not interval 0 but 1, etc.

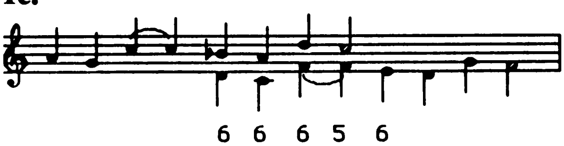
Example 11a.

-	A	G	C	C	B	A	D	C	C
D	5	4	7	7	6	5	8	7	7
C	6	5	8	8	7	6	9	8	8
F	3	2	5	5	4	3	6	5	5
F	3	2	5	5	4	3	6	5	5
E	4	3	6	6	5	4	7	6	6
D	5	4	7	7	6	5	8	7	7
G	2	1	4	4	3	2	5	4	4
F	3	2	5	5	4	3	6	5	5
F	3	2	5	5	4	3	6	5	5

Example 11b.



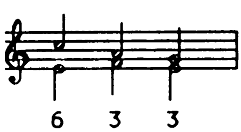
Example 11c.

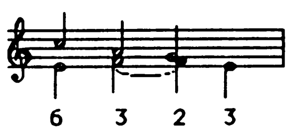


Example 12.

	C	A	G
E	6	4	3
F	5	3	2
E	6	4	3

	C	A	G
E	6	4	3
F	5	3	2
E	6	4	3





matrix shows what might not be apparent to the student who composed either example 11b or 11c, that there are two ways to align canonically the same two voices. Example 12 shows how a simple diagonal alignment on the left T-matrix can be elaborated as shown on the right matrix so that a series of consonances is ornamented by a 4-3 suspension. These last two examples show that the T-matrix models counterpoint in ways that make simple (computer-assisted) searches of the adjunct to more general algorithms and heuristics that attempt to automate the generation of legitimate traditional counterpoint.

Appendix A: Algorithm to Search for Transpositions of a Pcseg in another Pcseg.

Definitions and Initialization:

S = segment to search within
Z = segment to search for
z = #Z - 1
s = #S - 1
e = #S - #Z = s - z
zint = INT(Z)
F = the final segment, a transposition of Z
R = row
C = column
RR = opening row
E = T-matrix of S
N = order position of zint
M = order position of F

The Algorithm

```
1. RR = -1

2. R = RR + 1
       IF RR > e
       THEN
        STOP
       ENDIF
       N = 0
       M = 1
       RR = R
```

$F_0 = S_{RR}$

```
       C = R
```

```
3. IF C > s
        THEN
    GOTO 2.
        ENDIF

4. IF E<R,C> = zint_N
  THEN
        F_M = S_c
        M = M + 1
        IF M > e
        THEN
          F = T_n Z and F is included in S
          COMMENT: N = S_RR-S_O
          GOTO 2
        ELSE
    N = N + 1
          R = C
          GOTO 3
        ENDIF
  ENDIF

5. C = C + 1
  IF C > S
  THEN
        GOTO 2.
  ENDIF
  GOTO 4.
```