The Structure of First-Species Canon in Modal, Tonal and Atonal Musics

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It has been frequently and appropriately noted that serial composition has affinity with pre-tonal polyphony. Certainly both practices take the combinations of lynes of pitch-classes as their structural frame, and a distinct lack of harmonic teleology heightens the connection. But since fundamental research into modal counterpoint has been rare in the twentieth century, notwithstanding Serge Taneiev's brilliant counterpoint treatise of 1909 and a few articles by American theorists,[1] there have been few studies that relate the two practices in any deep structural way. This paper attempts to bridge this pre/post-tonal gap by studying first-species canons, primarily at the delay of one note, using a single methodology reinterpreted as necessary to accommodate modal, tonal and post-tonal sounds and concepts.[2]

Admittedly, the limitation to studying first-species canon restricts the scope of this paper. For, aside from simple four-part chorales, note-against-note textures are hardly a norm in music literature. Nonetheless, the study of first-species polyphony addresses both primary and advanced concerns in composition and analysis. From the Renaissance to the present, the note-against-note contrapuntal ideal has been asserted as fundamental to the study of both modal and tonal counterpoint and, in the 20th century, it has provided a methodological base for doing tonal (Schenkerian) analysis. Canon, on the other hand, is an advanced topic in modal counterpoint, taken up only after all five species have been mastered. This contrasts with post-tonal serial music where canon is primary, since all aspects of serial polyphony may be considered

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[1]See Serge Taneiev, *Convertible Counterpoint* (Moscow, 1909).

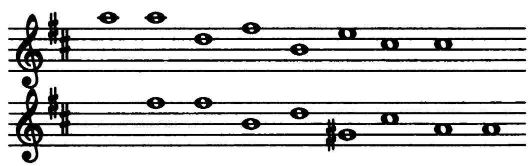
[2]Robert Gauldin has independently proposed a set of paradigms for composing and analyzing modal canons some of which are similar to our principles for generating canons. See his "The Composition of Late Renaissance Stretto Canons," *Theory and Practice*, forthcoming.

Example 1a. Stretto from a Bach Fugue.

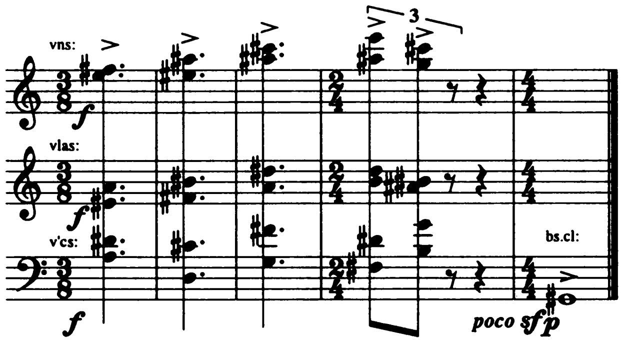
Canon at the third below:

—WTC II, Fugue in D major, mm. 45-7.

Underlying first-species canon at one beat delay:



Example 1b. Rotational Array from Stravinsky's "Huxley" Variations.



—Variations, mm. 137-141

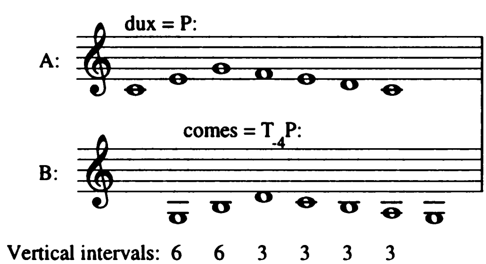
Underlying six-part, first-species pitch-class canon:

						Rotational Array											
P:	0	1	2	t	8	3	0	1	2	t	8	3					
T_{10}P:		t	e	0	8	6	1	t	e	0	8	6	1				
T_6P:			6	7	8	4	2	9	6	7	8	4	2	9			
T_7P:				7	8	9	5	3	t	7	8	9	5	3	t		
T_8P:					8	9	t	6	4	e	8	9	t	6	4	e	
T_5P:						5	6	7	3	1	8	5	6	7	3	1	8

abstract canons at various intervals, inversions, retrogrades, and delays.

But despite the importance of first-species texture and canon in music pedagogy and analysis, it is difficult to find musical examples that combine both canon and one-to-one polyphony. First species canons find their only uses in the structural foundation for close canons in the strettos of fugues or in *vorimitation* of Baroque chorale preludes. Example 1a shows that a first-species canon at the third below and delay of one beat underlies a stretto in a Bach fugue. In serial music, rotational arrays of pitch-classes, when interpreted homophonically, form another example of first-species canon.[3] See Example 1b for a rotational array underlying the final measures of Stravinsky's *Variations*. The leading and following pcs (which do not occur in the work) show how such an array is identical to the middle segment of a pc canon. But despite their rarity in the literature, as we shall see, first species canons provide insight into the art of canon in older music as well as offer today's composers some interesting compositional resources.

Example 2. Canon at the fourth below and at a delay of 1 beat:



[3]See Robert Morris, "Generalizing Rotational Arrays," *Journal of Music Theory*, 32/1 (1990).

Consider Example 2. It contains a note-against-note canon a fourth below at the delay of 1 beat; I use the traditional terms *dux* for the leading voice, and *comes* for the following voice. The only vertical intervals are thirds and sixths, and the only melodic intervals are the ascending third and the descending second. The serial labels P and $T_{-4}P$ are used to denote the relation between the voices.

Note that, as usual, we use integers to denote the tonal intervals. For melodic ordered intervals: 1 is a note repetition, -4 a descending fourth, +3 (or 3) an ascending third, and so forth. Distinctions between major and minor or augmented and diminished intervals are not needed at this point. In two-voice counterpoint vertical intervals are ordered, from the "lower" voice to the "upper" voice. Therefore, vertical intervals (that is, intervals produced by two simultaneous notes, each from a different voice) are negative when voices are crossed. So: 1 is a unison, 2 is a second, 3 is a third, and so on when voices are uncrossed; when voices are crossed a vertical third is given by -3, a vertical fifth by -5, et cetera. As is well-known, the traditional interval names produce problems when adding intervals. For instance, a third and a sixth "sum" to an octave, but 3 + 6 = 9.[4] Despite this drawback, here we use the traditional names to promote comprehension.

As Example 3 shows, we can describe the features of any two-voice canon as an ordered quintuple:

(P, N, D, I, J)

where P is the name of the canon tune or subject, N is the pitch interval of entry between the *dux* and *comes*, D is the time delay, I is the set of vertical intervals between the voices, and J is the set of linear or melodic (ordered) intervals in the tune. As shown in Example 3, (P, -4, 1, {3, 6}, {+3, -2}) describes the canon in Example 2.

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[4]More problems arise when negative values are also employed. The following is the *rule for addition* of intervals when conventional interval names are used: Let x and y be intervals, with $|x|>=|y|$. If x = -y, then x+y is defined to be 1. Otherwise, x+y is defined as x+y-1 if y>0, and x+y+1 if y<-1.

Example 3: Canon descriptors.

General canon descriptor: (P, N, D, I, J)

P = canon subject
N = interval between *dux* and *comes*
D = time delay (in beats)
I = set of vertical intervals in canon
J = set of linear (melodic) intervals in P

Descriptor for canon in Example 2:
(P, -4, 1, {3, 6}, {-2, +3})

In modal and tonal music, the vertical intervals allowable for inclusion in set I are the same whatever the pitch interval or delay; these are the consonant intervals {1, ±3, ±5, ±6, ±8, ±10, ±12}. Similarly, the linear intervals suitable for this music (for inclusion in J) are drawn from the set {1, ±2, ±3, ±4, ±5, ±6, ±8}.[5] Intervals larger than 3 are context dependent; for instance, the melodic sequence +4, +4, is not allowed.

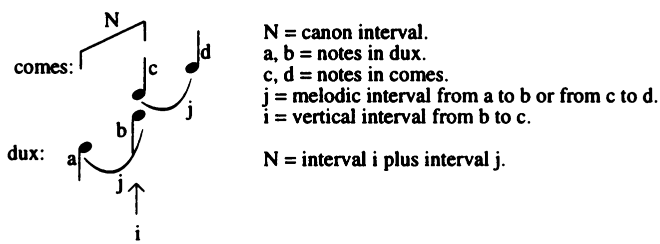
As any novice soon finds out, the limitation to these intervals imposes severe constraints on the production of correct counterpoint, especially in the composition of canons. These constraints, which we will eventually display via a graph, are dependent on N, D, I and J. Then, if a subject P can be derived by following a connected path on such a graph, P will generate a correct canon at N and D.

Consider Example 4a, the simplest canon fragment of two notes. The *dux* presents the notes a then b. This linear interval from a to b is called j. The *comes* starts its note c with b of the *dux* and then finishes with d; it also produces the interval j. Now, D is 1 and N is the interval from a to c (or b to d). The vertical interval i is the interval from b to c. From all of this, N equals the sum of intervals j and i. Since the legal values of i

[5]Note that in modal counterpoint, descending leaps of the sixth are discouraged. So +6 would be substituted for ±6 if we were to confine our study to modal polyphony.

are specified in tonal music, they control the values of j for a given N. Example 4b provides some examples of values for i and j where N is +5 and D = 1 that generate valid canon fragments, those that obey the rules of modal or tonal counterpoint. Note that some linear intervals (j) are ruled out by the choice of N. With N = +5, for instance, j cannot be +2, for i would have to be +4, a dissonance. Note also that i will be negative during crossed voices, when note b is higher than note c.

Example 4a. A minimal canon fragment.



Example 4b. Some values for j and i that generate valid canon fragments at N = +5.

To generate canons longer than that of Example 4b, we simply concatenate two linear intervals that satisfy the constraints of i, j, and N. Example 5a is a chart of all the possible successions of two intervals for N = +5. The melodic intervals, j, are -8, -6, -4, -2, 1, 3 and 5. These are given by the row and column heads of the chart; each cell of the chart, called a *case*, contains a canon fragment generated by the interval succession of its row interval, then column interval. (The case of the

Example 5a. Canon at the fifth above: the +5 canon system. Valid cases in half notes

Legend for invalid cases:

C = contour of line.

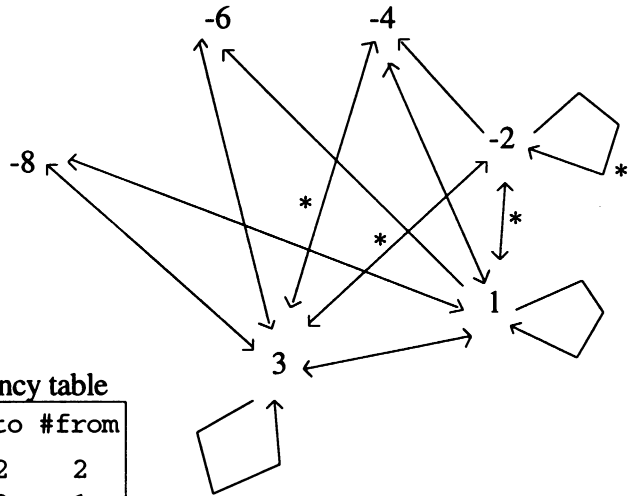
OC = overall contour of polyphony.

2P = parallel or contrary 5s or 8s.

NB: all cases on the bottom row are invalid since a 1 5 8 "tonal axis" is violated.

linear interval succession -4, 3 that generates a three-note canon at the fifth above is therefore found in the cell in the -4 row and the 3 column of the chart.)

Example 5b. Graph for canon at fifth above.



adjacency table

int	#to	#from
-8	2	2
-6	2	1
-4	3	3
-2	4	4
1	5	6
3	6	6
5	0	0

* = repeatable arrow

Of course, not all cases in the chart generate valid canons—canons that meet stylistic norms and constraints of 16th-century or 18th-century counterpoint. For instance, -4, -4 generates a highly untypical melodic sequence as well as parallel octaves. We therefore prune the chart according to the criteria given in the legend; we prune contours of lines or polyphony that are usually forbidden by the rules of counterpoint and, of course, parallel or similar octaves and fifths.[6] After pruning, only valid cases are left; these are written in half-note values while the invalid cases are written in quarter-notes. From the chart we learn that the melodic interval -6 can form a valid canon fragment if and only if it is followed by the interval 3. The same interval may be only preceded by the intervals 1 or 3. Such interval succession can be illustrated by the use of a graph as in Example 5b. The graph's nodes are the melodic intervals from which the canon subject is constructed. The graph shows every legal case from interval i_1 to i_2 on the chart, by an arrow connecting i_1 to i_2 on the graph. In other words, the arrows show which intervals may be concatenated to form a subject that automatically obeys the rules of counterpoint inscribed in the chart. Example 5b shows the graph derived from the chart in Example 5a. It indicates that the intervals -8, -6, -4, -2, 1, and +3, can be sequenced to compose a melody that will produce a correct canon at the upper fifth. Note that +5 is not on the graph because it was not involved in any valid cases. One traces a path following the arrows from interval to interval, never immediately taking the same two-way arrow in direct succession unless it is marked with an asterisk. The adjacency table in Example 5b tallies the number of different intervals to and from each of interval on the graph. Some canons derived from the graph are given in Example 6. The first of these follows the path -8, 3, -2, 3, 3, -4, 3 on the graph. The last, written by Robert Gauldin, is embellished and ends with a typical 16th-century cadence.

The chart and graphs of Example 5 form a *canon system* for N = +5. A change of N produces a canon system with different linear intervals. For example, the canon system for N = -2 allows only the linear intervals of -4, -2, 2, 4, and 5.

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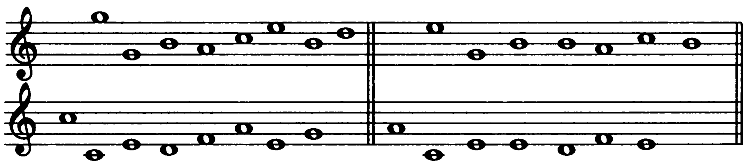
[6]Indeed, it would appear that the differences between modal and tonal canons could be defined by what criteria are used to prune such charts. However, the cases pruned from Ex. 5a are invalid in both modal and tonal polyphony. Of course, in a context not limited to first-species, two-voice canon the differences between the two styles (including each's own inconsistencies) would need to be explicitly distinguished.

Example 6. Canons generated by the +5 canon system.

6a. 6b.

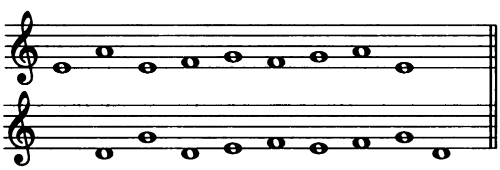


6c. Embellished with cadence.

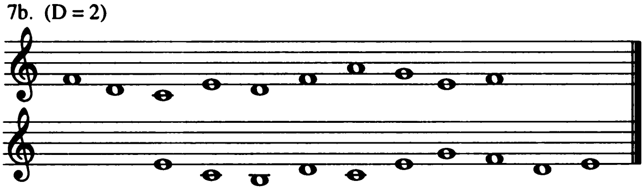
-Robert Gauldin

Example 7. Canons generated by the -2 canon sytem.

7a.



7b. (D = 2)



Example 7 presents two canons from this system. Note that the latter has D = 2.

The table in Example 8 gives the melodic intervals for all canon systems for N = -8 to +8. The column heads give the values of N. The row heads provide the legal, that is, consonant vertical intervals. Each column provides the melodic intervals that will yield the vertical intervals of the row heads on the left. This means that the content of the table position at the intersection of row x and column y added to the content of row head x will yield the content of column head y. Only melodic intervals that are permitted in traditional counterpoint are given; thus, ±7 and all intervals larger than 8 are omitted. Looking at the column under N = +5, we find the melodic intervals of the N = +5 canon system. However, one of this column's intervals is not in the graph of the N = +5 canon system, namely +5; it was ruled out in the chart in Example 5a because it consistently produced invalid cases. Thus the melodic intervals in the table may be pruned from a particular canon system for stylistic reasons. Nevertheless, the table allows us to compare the melodic intervals in various canon systems. For instance, the intersection of the melodic intervals for N = 3 and N = 5 produces the set of melodic intervals: {-8, -6, -4, 1, 3, 5}. Thus, melodies that only use these intervals may satisfy both canon system N = 3 and canon system N = 5. The table also indicates that canon systems with inverse Ns have inverse melodic intervals, a point I will amplify later in the discussion of *canon groups*.

In any case, Example 8 highlights the fact that each canon system has its own unique melodic signature. This is how a melody with a particular canonic potential can be immediately recognized. For instance, as we have seen, melodies that sequence intervals -8, -6, -4, -2, 1, and/or +3, will produce a correct canon at N = +5, at the fifth above. By way of contrast, melodies that sequence intervals -4, -2, 2, 4, and/or 5 will produce a correct canon at N = -2, the second below. In addition, the canon system graphs generate motivic signatures for each system. For instance, for N = +5, the interval sequences <3, -2> or <-2,-2> generate correct canons; for N = +2, <4,-2>

Example 8. Table of melodic intervals for canon systems, N = -8 to +8.

	−8	−7	−6	−5	−4	−3	−2	1	2	3	4	5	6	7	8
12												−8		−6	−5
10										−8		−6	−5	−4	−3
8								−8		−6	−5	−4	−3	−2	1
6						−8		−6	−5	−4	−3	−2	1	2	3
5					−8		−6	−5	−4	−3	−2	1	2	3	4
3			−8		−6	−5	−4	−3	−2	1	2	3	4	5	6
1	−8		−6	−5	−4	−3	−2	1	2	3	4	5	6		8
−3	−6	−5	−4	−3	−2	1	2	3	4	5	6		8		
−5	−4	−3	−2	1	2	3	4	5	6		8				
−6	−3	−2	1	2	3	4	5	6		8					
−8	1	2	3	4	5	6		8							
−10	3	4	5	6		8									
−12	5	6		8											

Row heads: traditional consonant vertical intervals

Column heads: values of N.

Table entries: Each column gives the melodic intervals that will generate the harmonic intervals of the row heads in canon system N.

will generate a correct canon. So when we hear a note sequence of intervals <3,-2> we know it can be used in a subject for a canon at the fifth; and when we hear a <4, -2> sequence, we know it can be part of a subject for a canon at the second.

The canons of Examples 6c and 7b suggest paths for further inquiry. First, what of canons with D > 1? Second, how can one embellish first-species canons to make them more musically serviceable? Third, how is dissonance treatment and ornamentation accomplished within the graph methodology? While I do not have enough space to address these questions completely, it should be understood at the outset that they are highly intertwined. The process of contrapuntal ornamentation or diminution in effect divides the beat so that a canon originally at a delay of X beats, if suitably renotated, will be at a delay of at least twice X beats.

If one would like to generate a first-species canon without ornamentation where D is 2 or greater, there are two choices. The first is necessary but not sufficient. One can interleave two canons alternating them beat by beat to produce a canon with D = 2. In such a canon, taking every other beat will yield a independent canon at D = 1. Example 7b was composed using this method. Since beats in each of the two generating canons are not actually adjacent in the resulting canon, some of the cases originally pruned from the chart for stylistic reasons might be reinstated in this environment. On the other hand, care must be taken to make sure that the moves from one canon stream to the other, between adjacent notes in the resulting canon, obey contrapuntal and melodic rules. I concede that such contingencies make this method a bit dicey for generating tonal canons; but in the case of post-tonal canons, where rules of voice leading and dissonance treatment are not at issue, this method is a perfectly valid means of generating canons for arbitrarily large values of D.

A better and wholly general method to generate tonal/modal canons when D is greater than 1 is to generate charts that sequence two or more intervals and list the concatenations of these sequences in the body of the chart. Once again, cases that deviate from stylistic conventions are pruned.

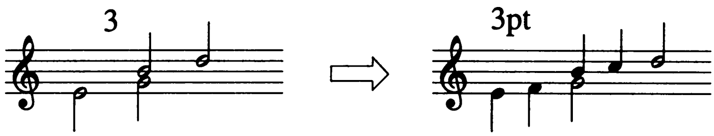
While this will generate graphs, the number of cases to consider grows very quickly so that resulting graphs become too cumbersome for humans to use. Of course, adjacency matrices and/or linked lists can substitute for graphs, but these data structures are more appropriate in computational studies.

Fortunately, methods that generate canons with D > 1 involving ornamentation and/or diminution are equally general but less delicate or complex. Here one has two basic options. One generates a canon from a graph, then embellishes it according to standard methods such as inserting passing and neighboring tones and the like. Such a two-pass method works adequately enough, but involves backtracking and models canon composition as two qualitatively different cognitive stages. However, there is a more elegant solution. One constructs a canon system not only from the melodic intervals, but from their transformations under standard ornamentation.

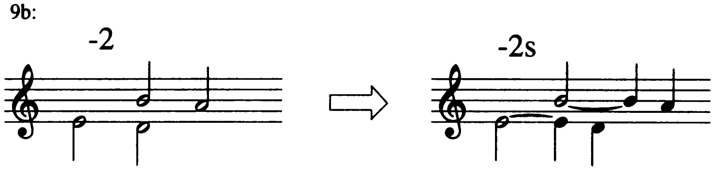
To illustrate this method, examine Example 9. In Example 9a we have the interval 3 written in the context of the canon system where D =1 and N = +5. We can transform the case on the left into the case on the right by inserting a passing tone in each ascending third. We call the transformed 3 "3pt". Similarly, in Example 9b, the interval -2, is transformed into a delayed "resolution" called -2s. Now we know from the graph of the +5 canon system in Example 5b, that interval 3 can be preceded by the intervals -8, -6, -4, -2, and 1 and that 3 can move to the same set of intervals. We then test pairs of intervals that begin or end with 3 (using the intervals just cited) substituting 3pt for 3 to see if any stylistic constraints for counterpoint or melody are violated. Since the test shows that every succession is legal, we can always substitute 3pt for 3 in the process of generating a canon from the graph. Thus, the transformation of 3pt for 3 is context-free, that is, valid in all cases of canon system N = +5. In the case of -2s, when we substitute -2s for -2 in all legal interval successions involving -2, not all cases yield legal contrapuntal textures. Example 9c lists all the cases; intervals that precede -2 are written as prefixes to -2s and their cases are illustrated on the left. Intervals that can follow -2 are written as suffixes to -2s and their cases

Example 9. Embellished cases.

9a:



9b:



9c:

("X" denotes cases pruned for stylistic reasons.)

are written on the right. Most of the cases in Example 9c are correct; only the two bottom cases on the right deviate from ordinary practice. These are marked with an X and indicate that the -2s transformation is context-sensitive. The first of these is the sequence -2s, 3 and produces a 4-3 suspension, which in many accounts of modal counterpoint is prohibited in two-voice textures; however, it should be remembered that two-voice 4-3 suspensions are tolerated in 18th-century polyphony, so this case could be retained. The recursive -2s, -2s case is incoherent since the first -2s makes it impossible for the second -2s transformation to find a half-note to delay. Nevertheless, I have written one way to implement the repetition of -2s which sufficiently disturbs the rhythmic stability of the case for it to be omitted. Looking over the rest of the cases, we see that we have implemented 7-6 and 6-5 progressions with the -2s transformation.

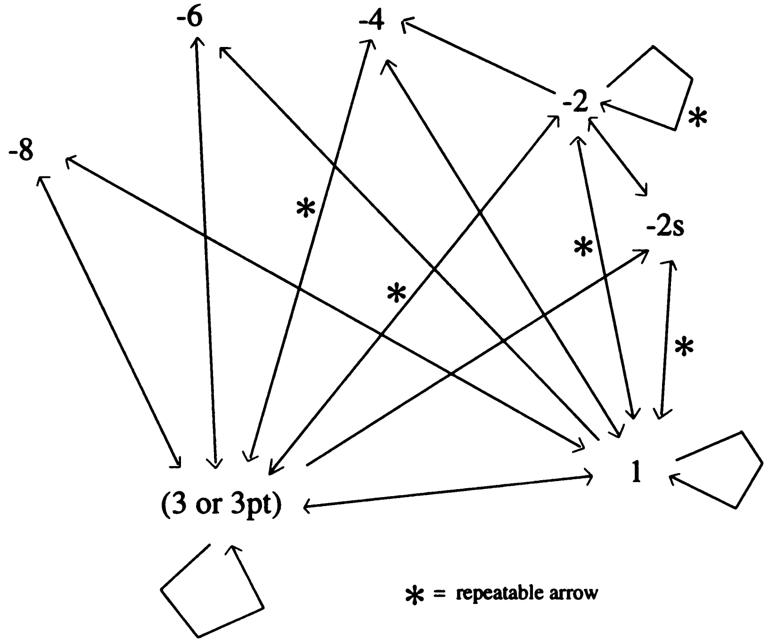
Now, we can add the remaining cases involving -2s and 3pt to the graph of canon system +5. The result is in Example 10a. The graph is interpreted as before and automatically generates correct canons with ornamentation. A canon generated from the graph is given in Example 10b.

Of course, we could have added 3pt and -2s as row and column heads in the original chart for the +5 canon system in Example 5a. The cases in Example 9 would then be cases in the chart and the pruning of -2s followed by 3 or itself would then be accomplished on the chart, rather than ex post facto as above. Other transformations, such as lower neighbor notes or runs of ornamental eighth-notes, à la third species, could be added as well.

As a transition to considering non-tonal canons, Example 11a illustrates basic symmetries among first-species canons and canon systems: how serial transformations affect a canon. In this discussion and throughout the rest of this paper, I shall use the term inversion only in its serial sense, to denote the transformation of a pitch or pitch-class onto its inverse. The exchange of voices, as in so-called "invertible counterpoint," will be called *voice exchange*.

Example 10

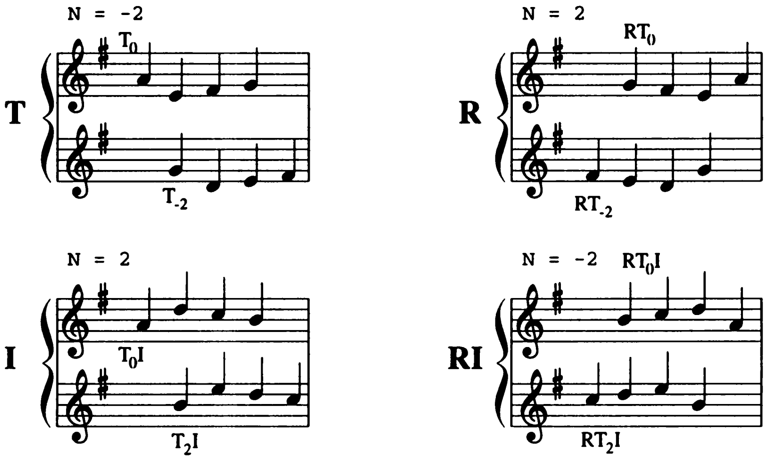
a. Graph for canon at fifth above with ornamentation.



b. Canon based on graph in Example 10a.

Sequence from graph 3 -2s 1 3 3pt -2s -2 -2

Example 11.
a. Serial transformations on canons.



b. The Canon Group.

P — subject
N — canonic interval
A — upper (lower) voice
B — lower (upper) voice

		R
T_0	A = P = dux B = T_NP = comes	A = RP = comes B = RT_NP = dux or A = RT_{-N}P = comes B = RP = dux
T_xI	A = T_xIP = dux B = T_{x-N}IP = comes	A = RT_xIP = comes B = RT_{x-N}IP = dux or A = RT_{x+N}IP = comes B = RT_xIP = dux

The canon fragment at the upper left is transformed under R, I, and RI. The *canon group*, the simple mathematical group that underlies such transformations is given in Example 11b. It indicates that: (1) every canon system for N is related to that of -N by inverting the linear intervals in the former's chart and graph; and that (2) under RI, the last four arguments of a canon descriptor remain invariant. This means canons related by RI are from the same canon system. In turn, this implies that first-species canon systems are not temporally teleological to within inversion. Lack of overall temporal teleology is typical of a good deal of 15th- and 16th-century polyphonic music; in contrast, Baroque music has such teleology built in due to its chord grammars and embedded harmonic/linear syntax. Of course, the introduction of ornamentation into canons in any style renders them temporally non-symmetric, for, after all, ornamentation delays and anticipates otherwise temporally symmetric musical events. This implies that transformational symmetry of the canon group of Example 11b does not invariably turn one correct ornamented canon into another. Composers will see this outcome as an instance of a general composition principle: the musical realization of a symmetric compositional design usually breaks its symmetry in the process of realization.

The extension of canon systems to non-tonal contexts is accomplished in two stages. First, the limits of tonal counterpoint and harmony are removed so that any pitch interval may be used as a melodic interval (in the canon subject) or as a vertical interval. Nevertheless this extension does not engage pitch-class relations as in serial music so that complementary and compound interval equivalence does not hold.

In the sequel, pitches, pitch-classes and intervals will be denoted as in most theories of atonal music, so that 0 denotes a unison, +7 denotes an interval of seven semitones, and so forth. We shall use t and e as for the pitch-classes 10 and 11, respectively. I shall also discontinue the use of the term "melodic interval" since the sequence of linear intervals in a non-tonal canon subject need not form a "melody."

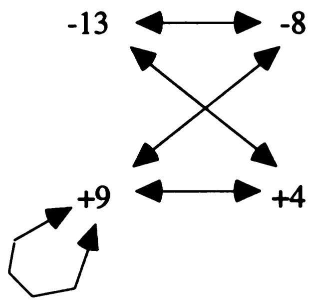
Example 12a. Pitch canon system (-2,{-11, -6, +6, +11}).

LEGEND:

O=proximate octave.

U="uninteresting" sonority/contour.

Example 12b. Graph for pitch canon system (-2, {-11, -6, 6, 11}).



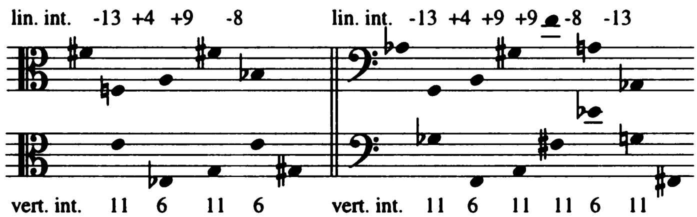
adjacency table

int	#to	#from
-13	2	2
-8	2	2
+4	2	2
+9	3	3

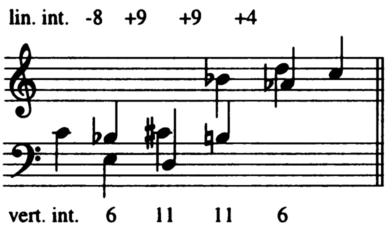
Examples 12a and 12b provide the chart and graph for a canon system with D = 1 and N = -2 with vertical intervals -11, -6, 6 and 11. The *comes* will follow the *dux* at one beat two semitones lower; the vertical intervals will either 6 or 11 semitones wide. We call such systems *pitch canon systems*. Different pitch canon systems are denoted by the couple (N, I), where I is the set of ordered vertical intervals. The pitch canon system at hand is (-2, {-11, -6, 6, 11}). As in any canon system, the vertical intervals in set I, when subtracted from N, produce the set of linear intervals in the system. So the linear intervals are: +9 = N –(-11) or -2 –(-11); +4 = -2 –(-6); -8 = -2 –(6); and -13 = -2 –(11). The chart of this pitch canon system has the four linear intervals as row heads. As in tonal systems, certain successions of intervals on the chart may be ruled out; yet here such cases are pruned from the chart by composer fiat rather than tradition. In this case, proximate octaves and "uninteresting" sonorities and contours (for instance, whole-tone scales) are not permitted. Some canons derived from this system are given in Example 13, the last of which is elaborated into a more typical 20th-century surface. Note that the musical realization in Example 13c2 camouflages the canonic voices of Example 13c1 in order to articulate various sets of adjacent canon pitches as instances of

Example 13. Canons at -2 below.
Vertical intervals = {-11, -6, 6, 11}.

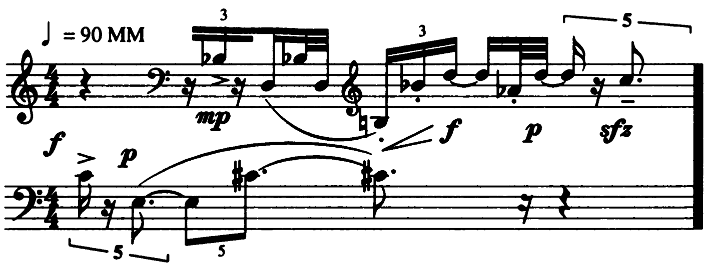
a. b.



c1.



c2. Elaboration of c1.



(pitch-class) set-classes 3-3[014] at the beginning and 3-8[026] at the end.

The second stage of non-tonal extension involves the mapping of pitch to pc. Hence, the charts are no longer written on the staff but with pc integers. These systems are called *pc canon systems*. The voices in the canons they generate are properly called *lynes*. (A lyne is an uninterpreted string of pitch-classes. For a lyne to be heard as a "voice," its pcs must be interpreted in some coherent global way, such as in the same register, in a given instrument, at a certain loudness, and so forth.) Example 14 provides a chart, graph, and pc canon for a pc canon system. Here the linear intervals are ordered pc intervals. The reader will note, that unlike in pitch canon systems, the vertical intervals (set I) are *unordered* intervals. I shall use the standard term *ic* (for *interval-class*) to stand for unordered pc intervals. (One could continue to use ordered intervals to model voice crossing, but I do not, if only because register is undefined among pitch classes.) The linear ordered intervals are derived by subtracting each member of each vertical ic from N.

While a pc canon system can be constructed directly as described above, a system can be derived from a pitch canon system as well. The pc canon system of Example 14 is generated from the pitch canon system in Examples 12 and 13 simply by taking the former system mod 12: N changes from -2 to t and the intervals -11, -6, 6, 11 are respectively mapped to the intervals 1, 6, 6, e, members of ic1 and ic6. Note that the set of vertical intervals has only three distinct members since the pitch intervals -6 and +6 both map to pc interval 6. This is the reason the chart for this pc system measures 3 by 3, instead of 4 by 4.

Certain cases in the chart of Example 14a are pruned according to roughly the same criteria as before; only hexachords of the set-class (SC) 6-2[012346] are chosen. Thus the resulting canons will be saturated by this hexachordal sonority. The graph is symmetric, allowing any sequence without immediate repetition of the linear pc intervals 4, 9, or e.

Example 14.

a. Pc canon system (t, {ic1, ic6}).
(Pc version of pitch canon system (-2, {6,11}) in Ex. 12a.)

e:	t 9 0 e SC(4-1)	e, e **D** *t 9 8* *0 e t* SC(5-2)	e, 4 t 9 1 0 e 3 SC(6-2)	e, 9 t 9 6 0 e 8 SC(6-2)
4:	t 2 0 4 SC(4-21)	4, e t 2 1 0 4 3 SC(6-2)	4, 4 **U** *t 2 6* *0 4 8* SC(6-35)	4, 9 t 2 e 0 4 1 SC(6-2)
9:	t 7 0 9 SC(4-10)	9, e t 7 6 0 9 8 SC(6-2)	9, 4 t 7 e 0 9 1 SC(6-2)	9, 9 **S** *t 7 4* *0 9 6* SC(6-23)

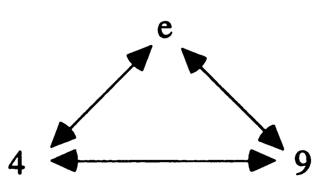
LEGEND

D = duplicated pcs.

U = "uninteresting" SC.

S = singular hexachordal SC.

b. Graph for pc canon system (t, {ic1}, {ic6}).



adjacency table

ic	#to	#from
4	2	2
9	2	2
2	2	2

c. A canon generated by the pc canon system (t, {ic1, ic6}).

Intervals:		e	9	4	e	4	e	
Dux:	0	e	8	0	e	3	2	
Comes:		t	9	6	t	9	1	0
Vert. ics		1	1	6	1	6	1	

The symmetry of any canon system graph is guaranteed so long as the composer does not prune any cases from the system's chart, so that all linear intervals are context free. After pruning, however, the intervals on the graph are context sensitive. Graph symmetry ensues if the composer prunes sets of cases that are symmetric around the main diagonal of the chart; that is, the case in the xth row and yth column must be pruned if and only if the case in the yth row and xth column is pruned. The symmetry of the present graph results from deleting cases along the main diagonal of the chart in Example 14a.

Pc canon systems have different properties than pitch canon systems. Among these is the underlying group structure relating pc canons. While registral interchange (i.e. double counterpoint) does not in general guarantee correct contrapuntal relations between voice pairs in tonal and modal polyphony (especially when the voices are in canon), in pc canon systems ics do not change when the "bottom" and "top" lynes are exchanged. We therefore compose a *pc canon group* of four transformations: T_0, RI, voice exchange, and RI with voice exchange. The pc canon group preserves N. This contrasts with the canon group of Example 11b which includes R and I as transformations and does not preserve N. Four canons generated from a canon from the pc canon system (4, {ic1, ic6}) interrelated by the pc canon group are given in Example 15a. The exchange operation allows the linkage of the four canons to form *canonic followings*. These are two concatenated canons, linked so that the last note of the first canon's *comes* voice is played with the first note of the second canon's *dux*. We constrain the link so it preserves the vertical ics of its generating canons. However, canon linkage may not preserve N. As an example see Example 15b which links canons related by T_0I; note that the canon interval (N = 4) changes to its inverse (8) after the link. To get canon followings that preserve N, one can enlarge the set of four canons related by the pc canon group to 48 under pc transposition and choose canons that utilize a link interval from the vertical intervals of the pc

Example 15.

a. Four pc canons related by the pc canon group.

(no exchange)			with exchange		
N = 4			N = 4		
P:	0 3 6 4 9	= dux	T_4P:	4 7 t 8 1	= old comes
T_4P:	4 7 t 8 1	= comes	P:	0 3 6 4 9	= old dux
N = 4			N = 4		
RT_0IP:	3 8 6 9 0	= old dux	RT_8IP:	e 4 2 5 8	= old comes
RT_8IP:	e 4 2 5 8	= old comes	RT_0IP:	3 8 6 9 0	= old dux

b. A canonic following of two canons related by T_0I.

P: T_0IP:

0 3 6 4 9 0 9 6 8 3

4 7 t 8 1 8 5 2 4 e link ic = 1 (at ↑)

T_4P: ↑ T_8IP:

Note: the canon interval N changes from 4 to 8 after the link.

c. Canonic followings that preserve N = 4.

P: T_7P:

0 3 6 4 9 7 t 1 e 4

4 7 t 8 1 e 2 5 3 8 link ic = 6 (at ↑)

T_4P: ↑ T_eP:

P: RT_9IP:

0 3 6 4 9 0 5 3 6 9

4 7 t 8 1 4 9 7 t 1 link ic = 1 (at ↑)

T_4P: ↑ RT_1IP:

P: RT_2P:

0 3 6 4 9 2 5 8 6 e

4 7 t 8 1 6 9 0 t 3 link ic = 1 (at ↑)

T_4P: ↑ RT_6P:

P: RT_4IP:

0 3 6 4 9 7 0 t 1 4

4 7 t 8 1 e 4 2 5 8 link ic = 6 (at ↑)

T_4P: ↑ RT_8IP

canon system. Some canonic followings that preserve N are given in Example 15c.

Example 16. Number of set classes that can comprise the pc content of pc canons.

n	m	# of set-classes available for two-voice canons.	# of set-classes available for three-or-more-voice canons.
3	12	5	5
4	29	13	13
5	38	15	15
6	50	30	33
7	38	30	31
8	29	25	29
9	12	11	12

n: Set class cardinality.
m: Number of set classes containing pcsets of cardinality n.

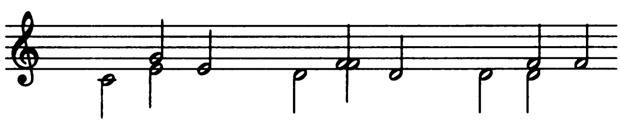
Since, allowing for pc repetition, there are seven ics and 12 ordered intervals, there are 1,524 distinct pc canon systems.[7] This contrasts with the 14 or so modal/tonal canon systems and the thousands of pitch canon systems available. Since we have used set-classes formed by two or more adjacent pcs in corresponding order positions of the *dux* and *comes* as a criterion for pruning certain linear intervals from a pc canon system, it is interesting that some set-classes cannot be found adjacently in any canon from any canon system, for any values of D and N in any number of voices. This follows from the fact that a pc canon's content is the "transpositional combination" of the content of its *dux* voice and the union of pcs that begin each voice.[8] Since some set-classes cannot be

[7] $1{,}524 = (2^7 - 1) \times 12$. For 3 ics there are $21 \times 12 = 252$ pc canon systems; $C(7,3) = 21$.

[8] See Richard Cohn, *Transpositional Combination in Twentieth-Century Music* (Eastman School of Music, University of Rochester: Ph.D. diss., 1986). The appendices in Volume 2 contain useful tables of primes, idempotent SCs, and the like.

generated by transpositional combination—those designated "primes" and "semi-primes" by Richard Cohn—they cannot occur in any pc canon as adjacent pcsets. A table indicating the number of set-classes that can comprise the pc content of canons of two voices or more is found in Example 16. The table shows, for instance, that of the 50 hexachordal set-classes only 30 can be produced by the pcs of a two-voice pc canon. Further, Cohn's "idempotent" set-classes allow certain pc canons to exhaust only a subset of the aggregate—regardless of the number of voices—providing one confines their subjects to certain set-classes and uses only certain values for N.

Example 17. Fragments from a canon by inversion at the fifth above.

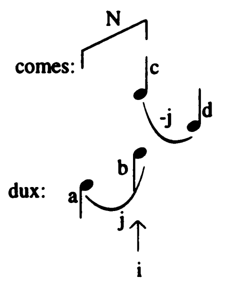


melodic intervlas in dux:	+3	+3	1
vertical intervals	3	1	3

Up to now, we have only considered canons by transposition. What about canons by inversion? These canons have their *dux* followed by the *comes* transformed under inversion and transposition. Unfortunately, there seems to be no way to generate inversional canons with the methodology developed herein. The transpositional canon systems considered so far are based on the condition that the melodic intervals in the *dux* and *comes* are mapped one to one to the vertical intervals between them. This is implied by the fact that N equals the melodic interval j plus the vertical interval i, as in Example 4a. Now consider the three cases in Example 17. These are canon fragments from a hypothetical canon by inversion at the fifth above at a delay of one beat. The first two cases show that the melodic interval is associated with two different vertical intervals while the first and last cases show that the vertical interval is associated with two different melodic intervals. Thus there is no mapping at all from the melodic intervals to the vertical intervals or vice versa. This means that a given melodic interval generates a particular vertical interval only if we know the first pitch of the interval. So it is impossible to generate a graph of interval successions that can be traced to generate all and only subjects for inversional canons. As a result, canon systems can be only tangentially related to George Perle's *Twelve-Tone Tonality*,[9] whose "cyclic sets" are canons related under rotation, retrograde, transposition and inversion operations.

Example 18.

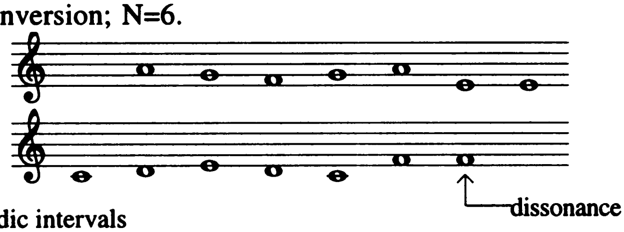
a. A minimal inversional canon fragment.



N = canon interval.
a, b = notes in dux.
c, d = notes in comes.
j = melodic interval from a to b
-j = melodic interval from c to d.
i = sum of b + c.

N = i - j if j > 0
N = i - (j +2) if j < -1

b. Canon by inversion; N=6.



j = melodic intervals

+2	+2	-2	-2	+4	1

i = harmonic sum

7	7	5	5	9	6

N = i + (-j) = 6

[9]See George Perle, *Twelve-Tone Tonality* (Berkeley: University of California Press, 1977).

Nevertheless, in the interest of theoretic closure, we can model canons by inversion using sums of pitches. Example 18a analyses a canon fragment from a tonal/modal canon by inversion at interval N. Pitches a to b form the interval j, and pitches c to d form the inverse interval -j. The variable i is the sum of the simultaneous notes b + c. Finally, N = i - j. Example 18b provides a canon by inversion at the sixth. As the example shows, N = 6 and the sums of the simultaneous pitches minus the preceding interval in the *dux* equals N or 6. The incorrect—that is, dissonant—last simultaneity helps assure us that the model works for all sums and intervals, dissonant or not.

The formal study of canon systems and their visual representations as transition graphs conjures up associations with David Lewin's generalized musical intervals and transformations.[10] Certainly, the set of ordered pitch or pitch-class intervals and their concatenations forming intervals from the same set under addition which provides the context for a canon system is the very GIS which has motivated Lewin's generalizations. Were the canon systems themselves instances of a generalized interval system, they would be matched by a powerful analytic methodology. Unfortunately, however, this is not the case for two reasons. First, the linear and vertical intervals of a canon system are not entities related by generalized intervals or transformations, but only by possible succession. Second, the intervals of the canon system graph nodes are not "things" as in Lewin's transformation networks, but simply equivalence classes of ordered pitch or pc pairs. Two pairs <a,b> and <c,d> are equivalent if b-a = d-c. This equivalence is coextensive with the concept of ordered pitch interval. Of course, the canon groups of Examples 11b and 15a can be modeled by a GIS and extended to model the invariances of invertible counterpoint for any number of voices.

In an important sense, we have been studying the relations between melodic structure and polyphonic opportunity.

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[10]See David Lewin, *Generalized Musical Intervals and Transformations* (New Haven: Yale University Press, 1987).

Example 19. An enigmatic canon.

a. Canon a 4: dedié A Monsieur Houdemann et composé par J. S. Bach.

b. Solution to the above.

(After *The Bach Reader*, edited by Hans T. David and Arthur Mendel, pp.111, 402.)

With more experience with tonal canon systems, we could develop the ability to perform musical parlor tricks to astound our colleagues, such as deftly transforming the enigma canon in Example 19a into its solution in Example 19b.[11] But there is more at stake here, for canon systems and other similar types of musical dependencies give us insight into the cognition of master musicians who accomplish complex musical tasks effortlessly in the real time of musical improvisation or listening. Specifically, the canon systems help to explain how J. S. Bach was able to perform the contrapuntal predictions described by Carl Philipp Emanuel Bach in a letter of 1774 to Johann Nicolaus Forkel:

> When he listened to a rich and many-voiced fugue, he could soon say, after the first entries of the subject, what contrapuntal devices it would be possible to apply, and which of them the composer by rights ought to apply, and on such occasions, when I was standing next to him, and he had voiced his surmises to me, he would joyfully nudge me when his expectations were fulfilled.[12,13]

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[11]The canon is transcribed into modern notation from Mattheson's *Der vollkommene Capellmeister* of 1739. The solution in Example 19b is given in *The Bach Reader*, revised edition, Hans T. David and Arthur Mendel, eds. (New York: Norton, 1966), p. 402.

[12]*The Bach Reader*, p. 277.

[13] I would like to thank Daniel Harrison and Dora Hanninen for reading early drafts of this paper and offering suggestions for improvement. My thanks also go to Norman Carey who added a few fine touches and corrections.